Nektar++: a high-order finite element framework

David Moxey
Department of Aeronautics, Imperial College London

ICES Seminar Series, University of Texas, Austin
19th June 2014

Overview

- Motivation
- Nektar++: what is it?
- Some challenges
- A few results

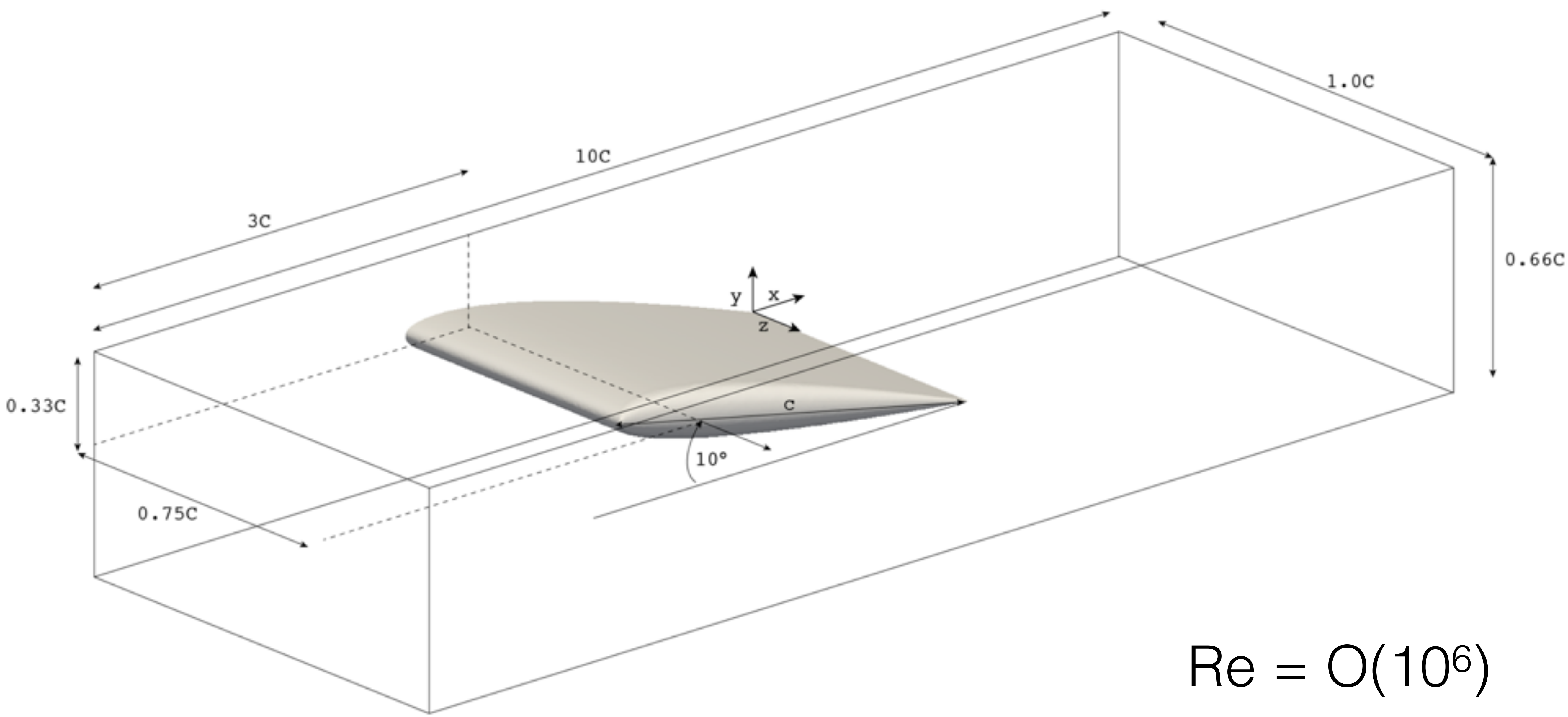
Motivation

Primary research goal is to investigate challenging external aerodynamics cases:

- High Reynolds numbers
- Complex three-dimensional geometries
- Large resolution requirements
- Transient dynamics

Using DNS + stabilisation

NACA 0012 wing tip



Re = O(10^6)

Difficult to capture transient effects with RANS

Nektar++ high-order framework

Open source framework for spectral(/hp) element methods:

- Dimension independent, supports CG/DG/HDG, manifolds
- Mixed elements (quads/tris, hexes, prisms, tets, pyramids)
- Solvers for (in)compressible Navier-Stokes, advection-diffusion-reaction,
- Parallelised with MPI, scales up to ~10k cores
- C++, heavily OO, git, cmake, boost + some other dependencies

http://www.nektar.info/
nektar-users@imperial.ac.uk

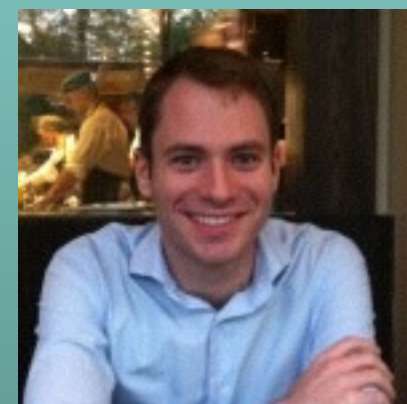
Joaquim Peiró

Nektar++ team

Spencer Sherwin

Alessandro Bolis

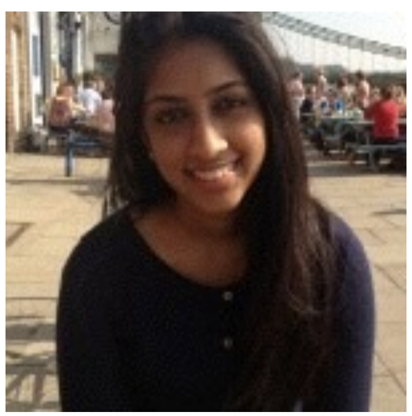
Chris Cantwell



Andrew Comerford

David Moxey

Hui Xu



Rheeda Ali

Paola Alpresa-Gutierrez

Dirk Ekelschot

Daniele de Grazia

Bastien Jordi

Jean-Eloi Lombard

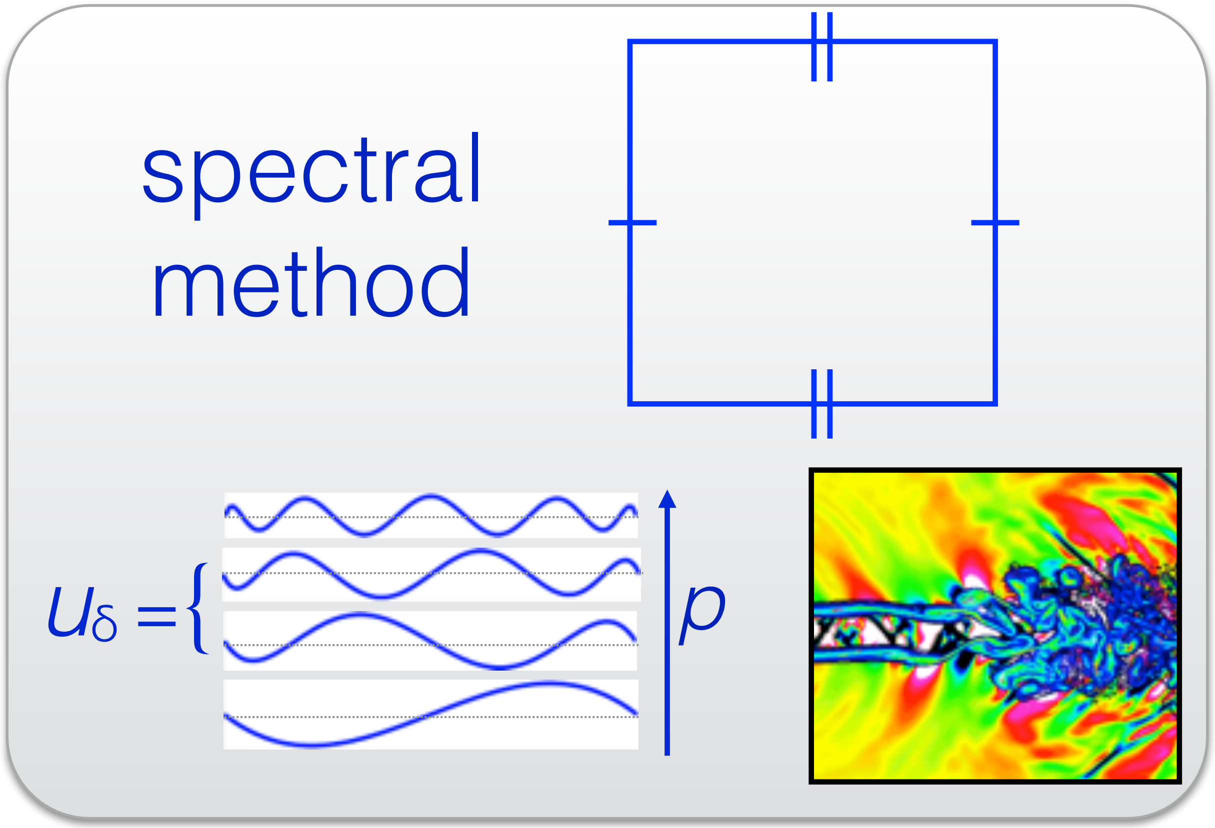
Gianmarco Mengaldo

Yumnah Mohamied

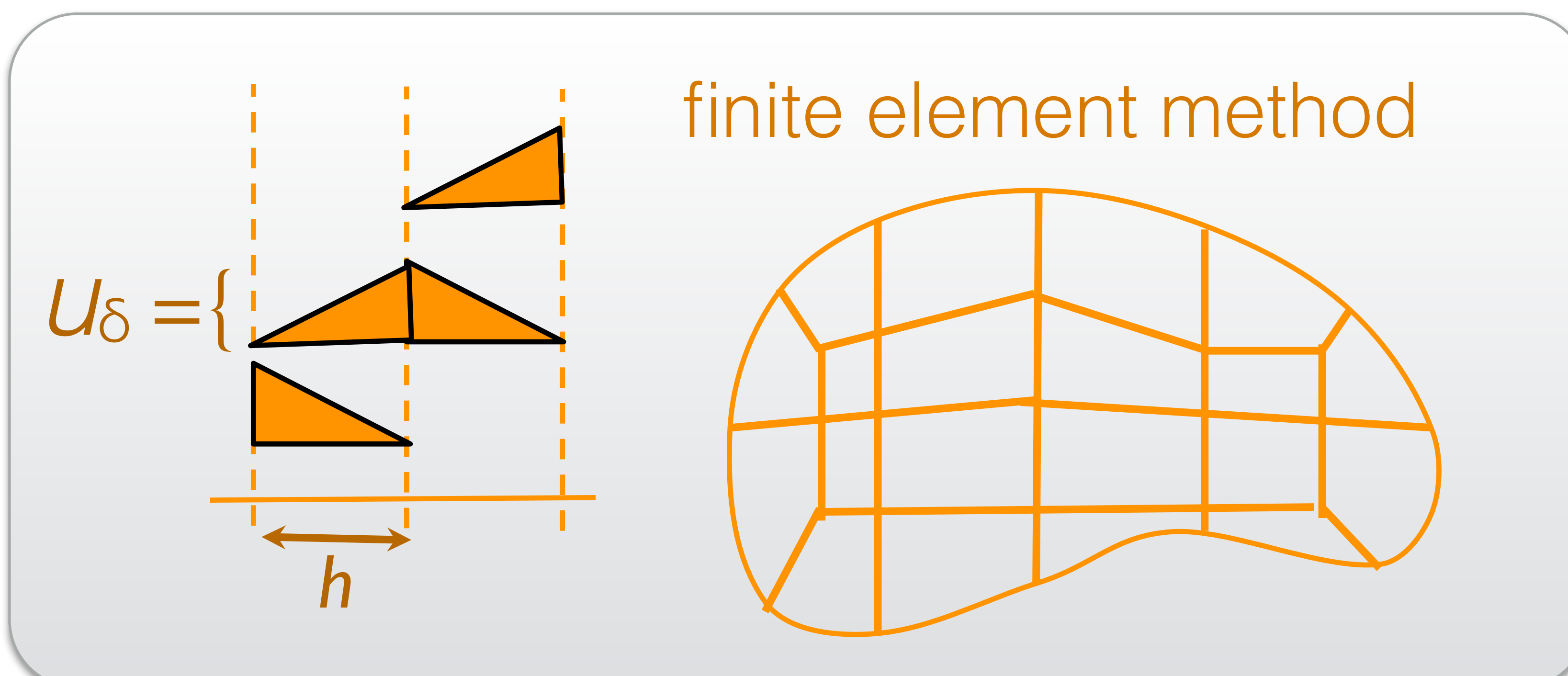
Rodrigo Moura

Gabriele Rocco

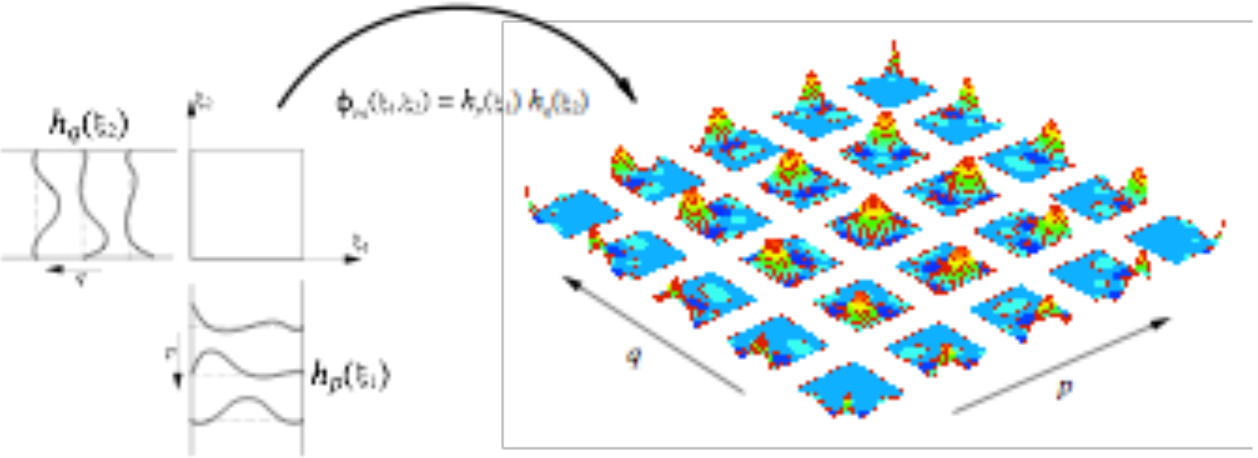
What is a spectral/*hp* element?



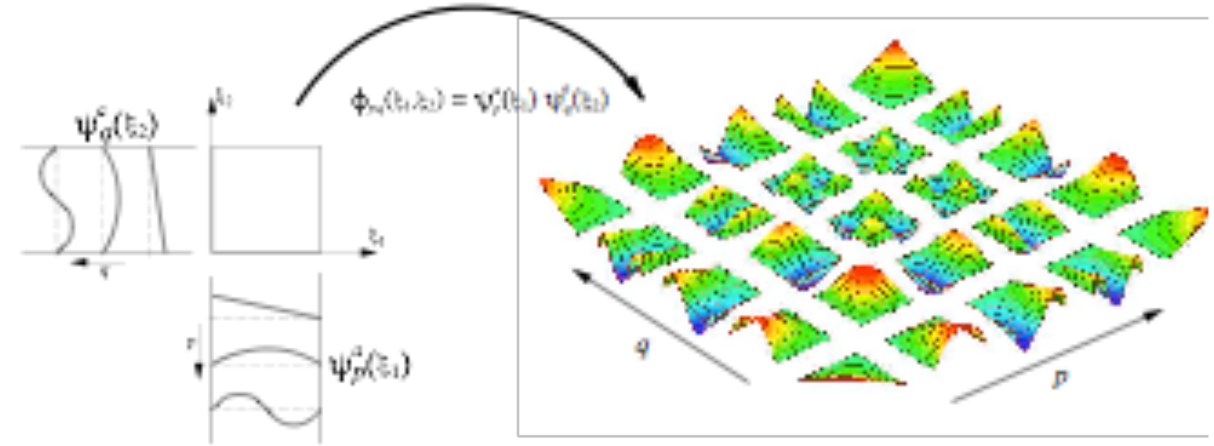
+



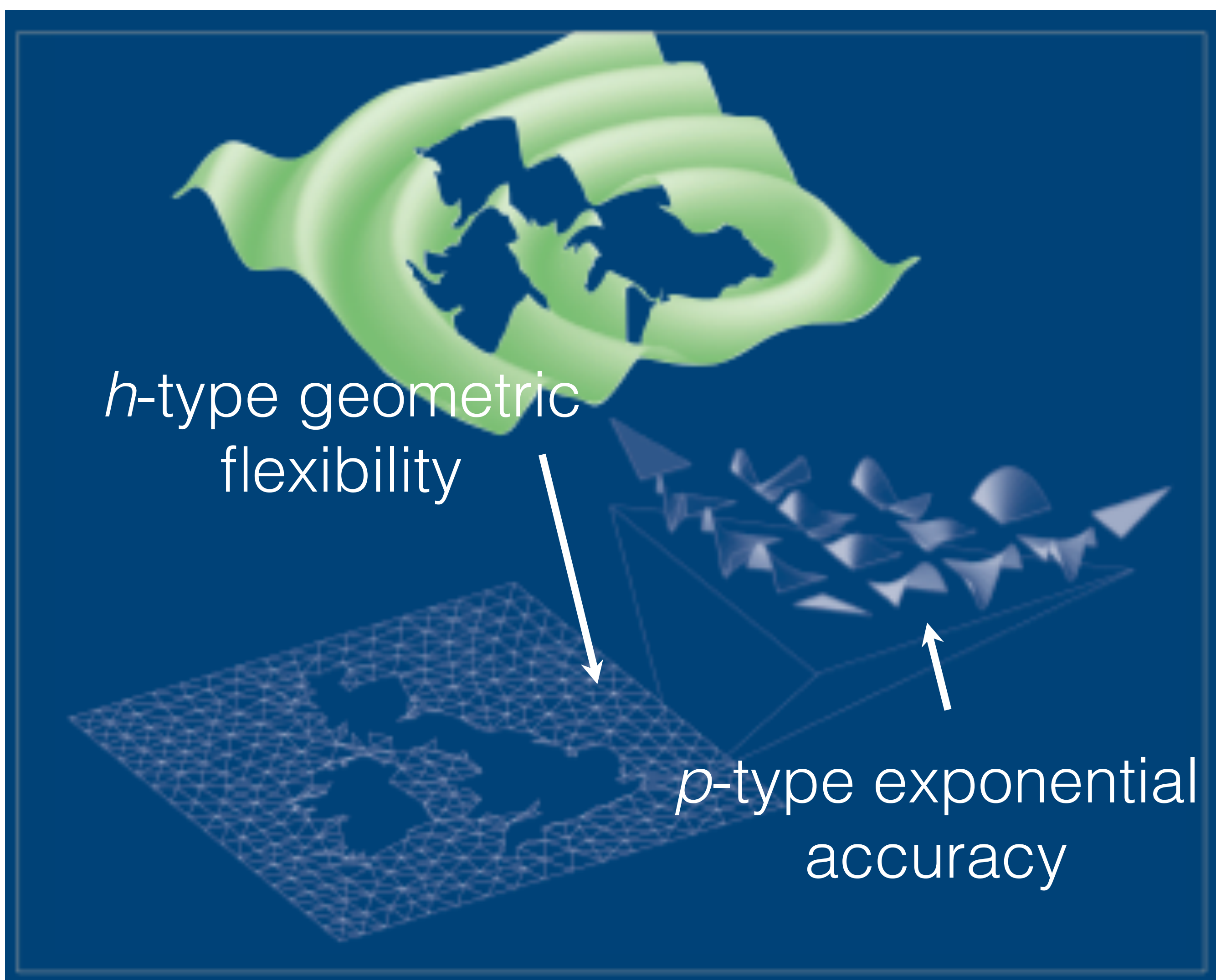
Spectral/hp element method



Spectral element method - nodal



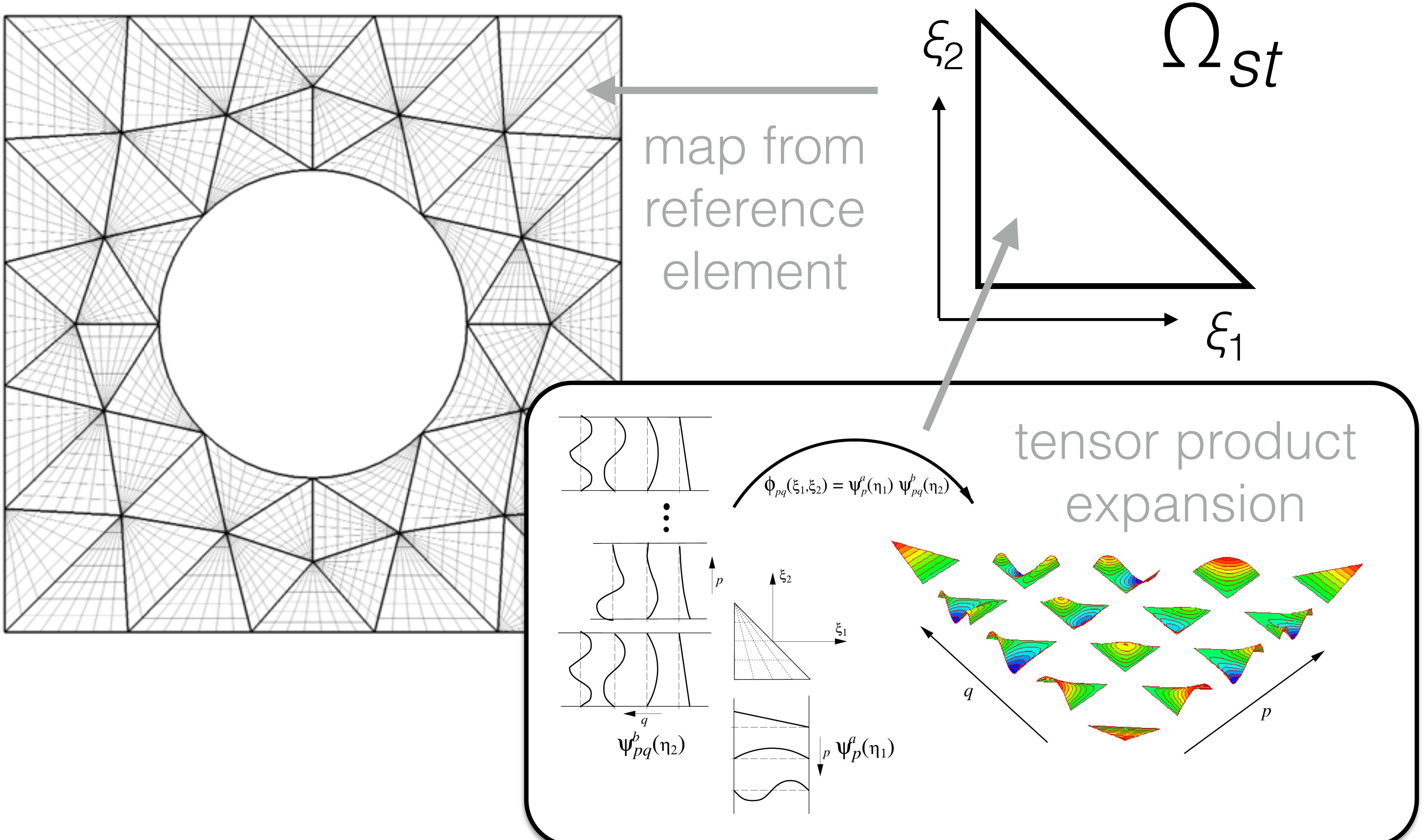
hp finite element - hierarchical



$$\text{Work} \approx C(h)p^{dim+1}$$

$$\text{Error} \approx K(u,p)h^{p}$$

Spectral/hp elements



Design

Consider the Helmholtz equation:

$$\Delta u + \lambda u = f$$

Written in weak form we obtain:

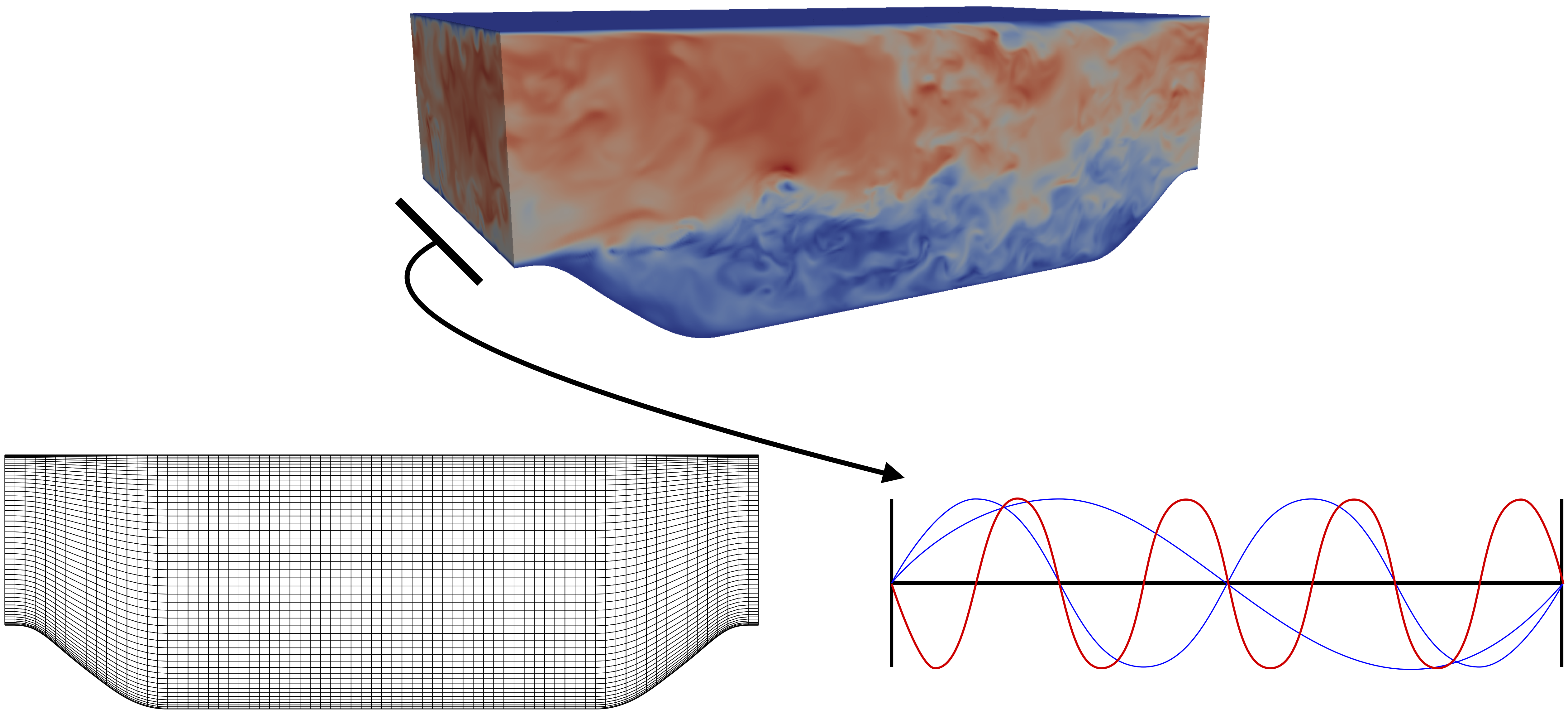
$$-(\nabla u, \nabla v) + \lambda(u, v) + (\nabla u, v)|_{\partial\Omega} = (f, v)$$

Expand in terms of local (per element) or global modes:

$$u_e^\delta = \sum_p \hat{u}_p \phi_p(x) \qquad\qquad u^\delta = \sum_i \hat{u}_i \Phi_i(x)$$

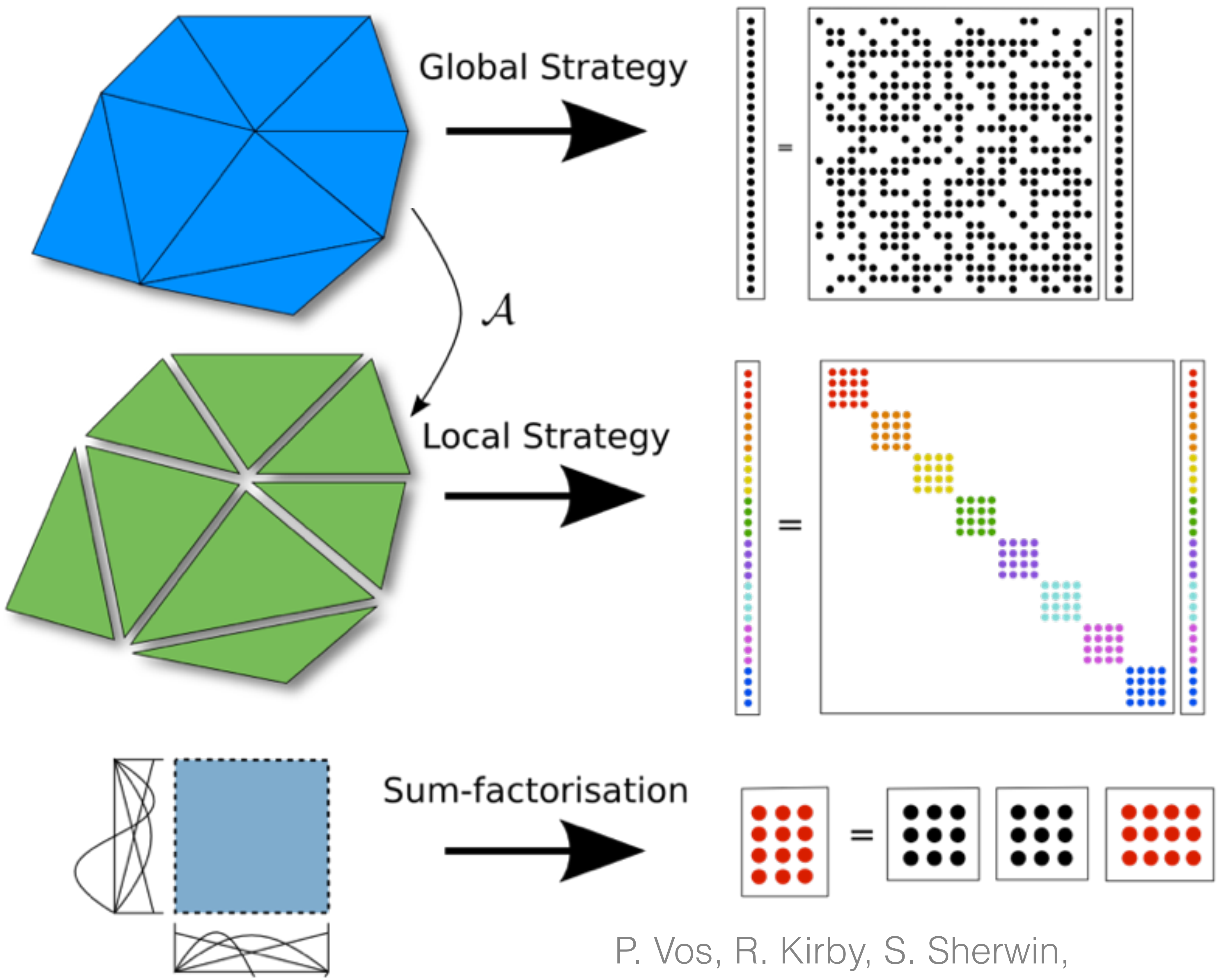
Symmetry

Can incorporate domain symmetry if possible: leads to wider range of parallelisation opportunities



2D Spectral element mesh + 1D Fourier expansion

Implementation strategies



P. Vos, R. Kirby, S. Sherwin,
C. Cantwell, S. Sherwin, R. Kirby, P. Kelly,

Some application areas

- Biological flows
  Y. Mohamied, A. Comerford

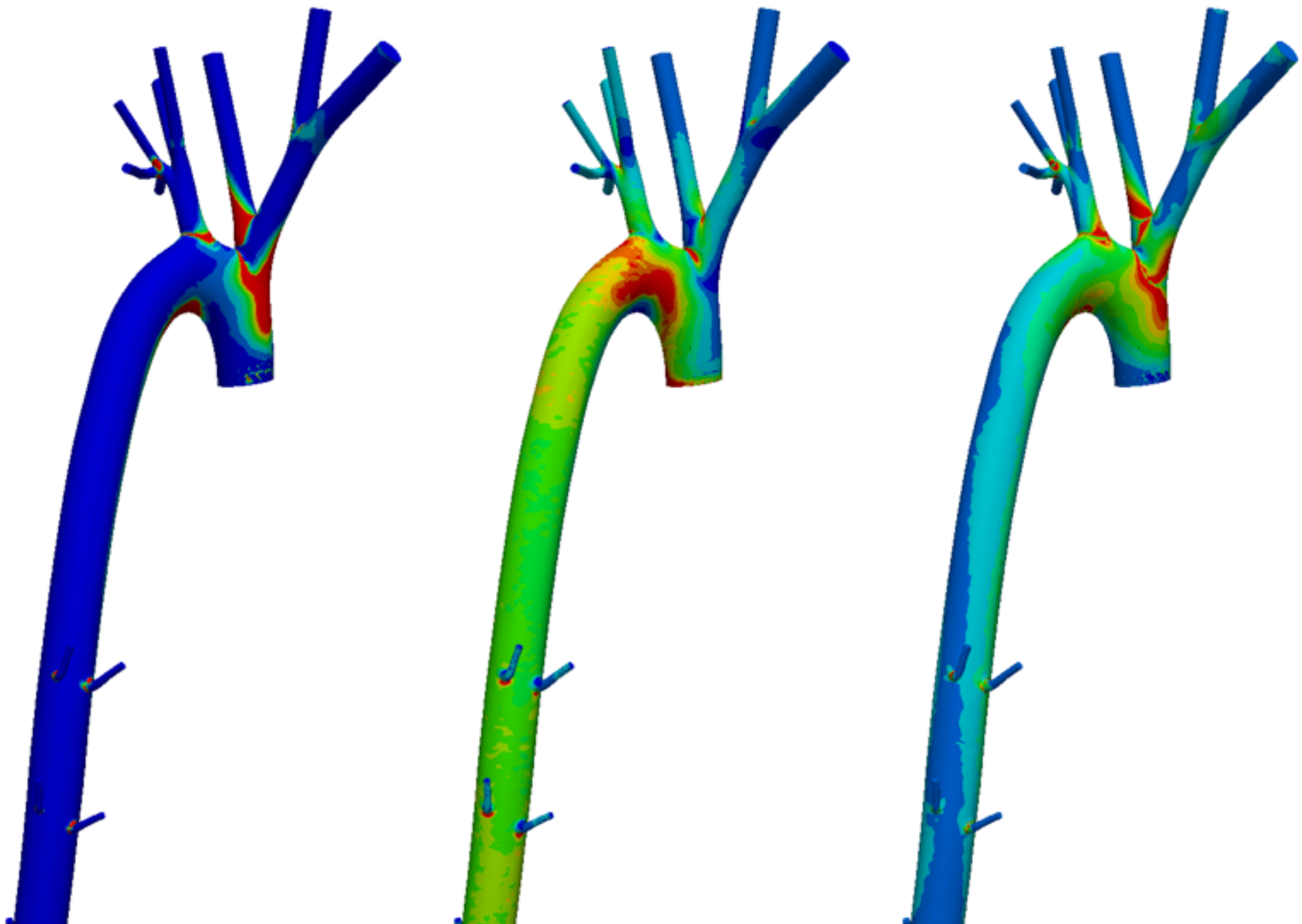
- Cardiac electrophysiology
  C. Cantwell, C. Roney, R. Ali

- Compressible flow
  D. Moxey, G. Mengaldo, D. de Grazia, D. Ekelschot, R. Moura

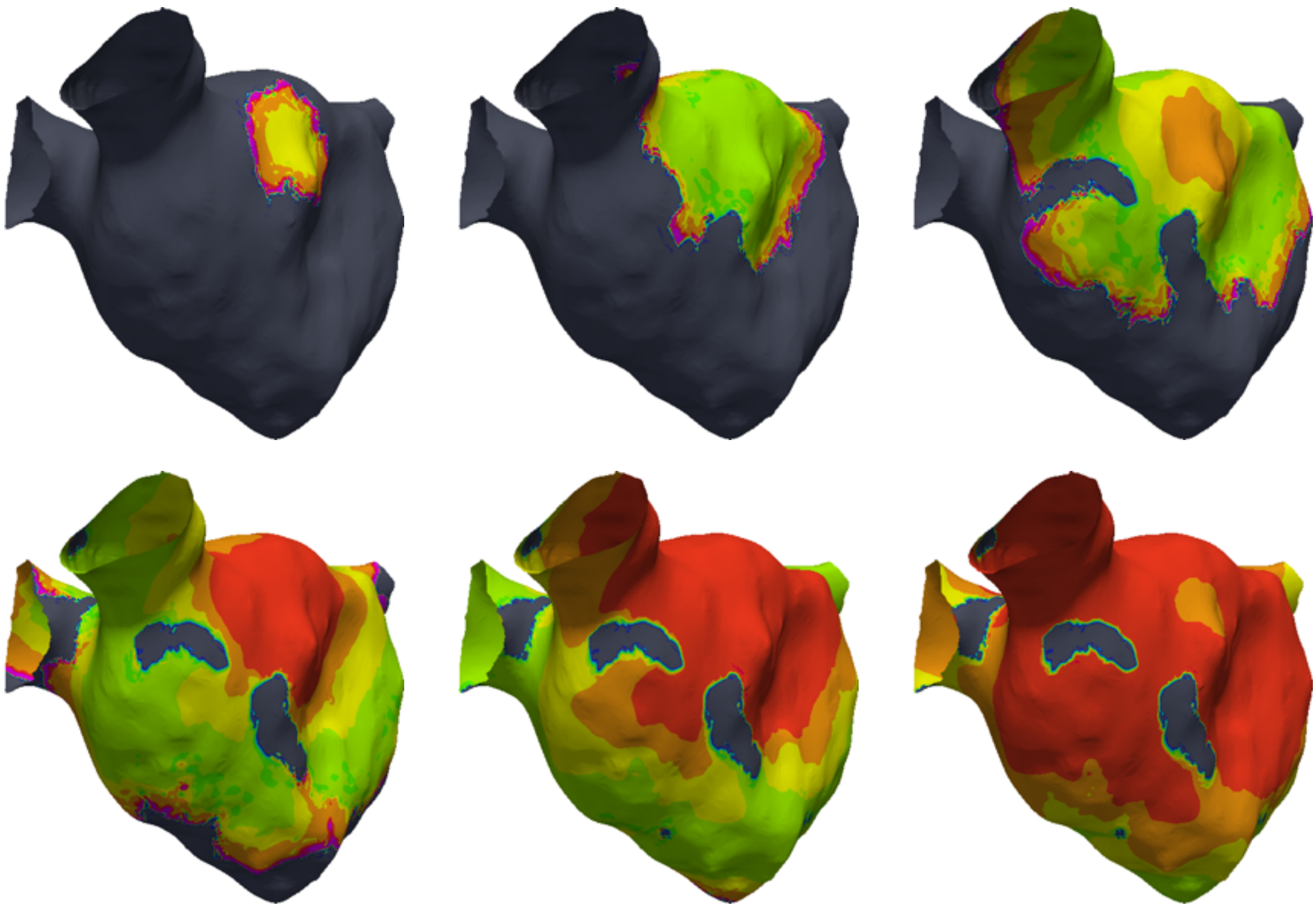
- **External aerodynamics**
  D. Moxey, J.-E. Lombard

Bioflows

Different time dependent shear metrics and understanding how these correlate with the initiation of atherosclerosis



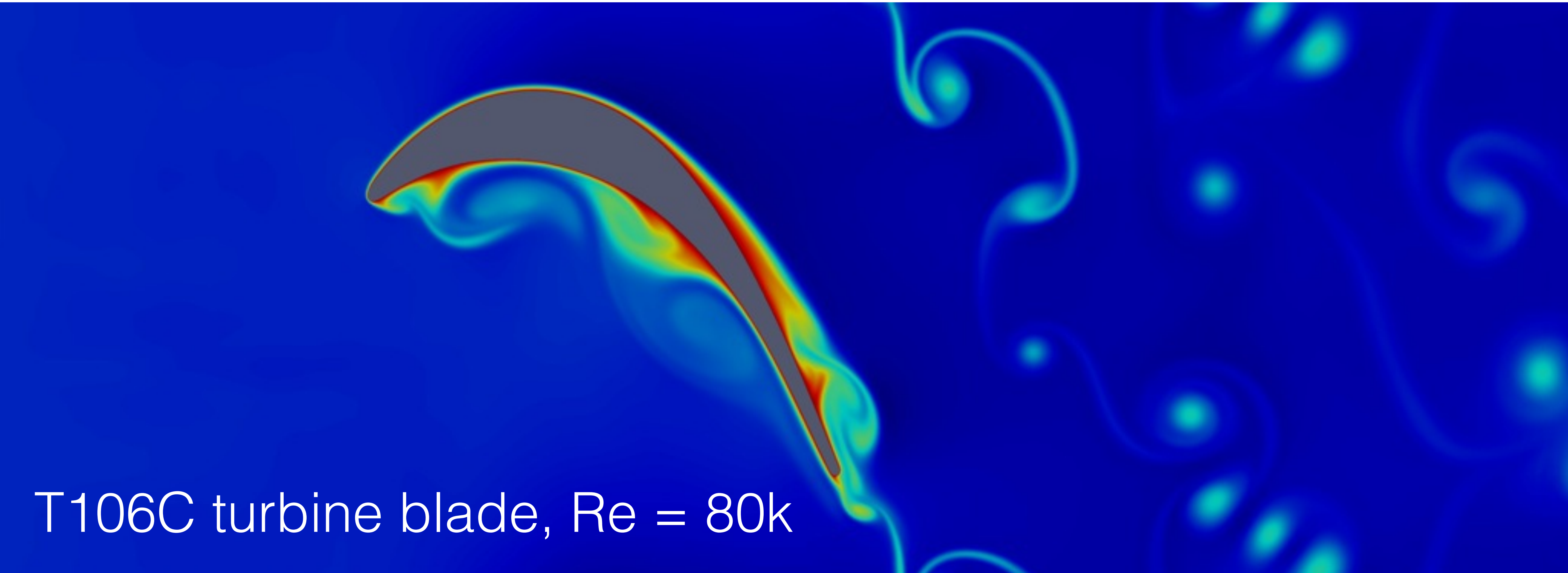
Cardiac electrophysiology



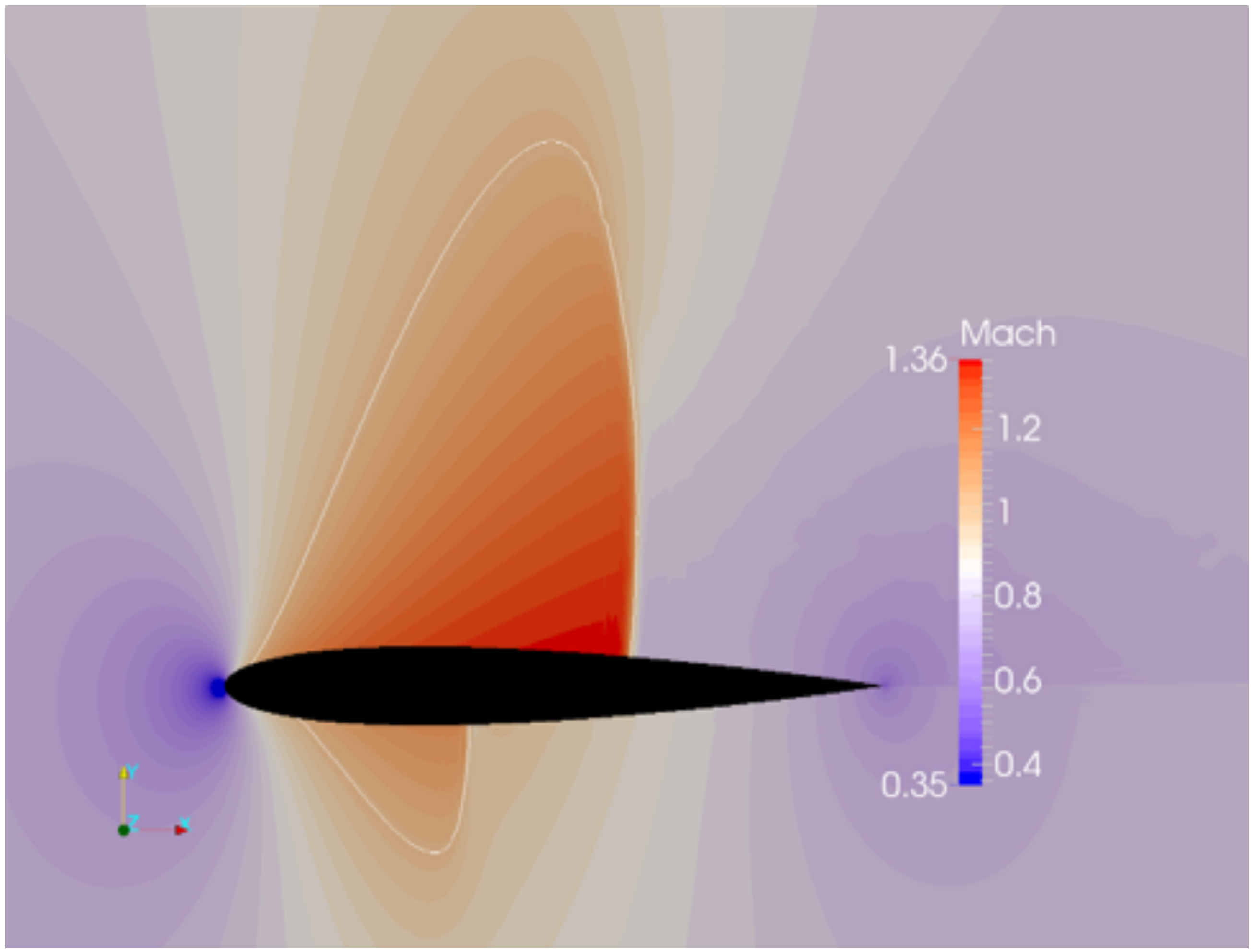
Compressible Flow

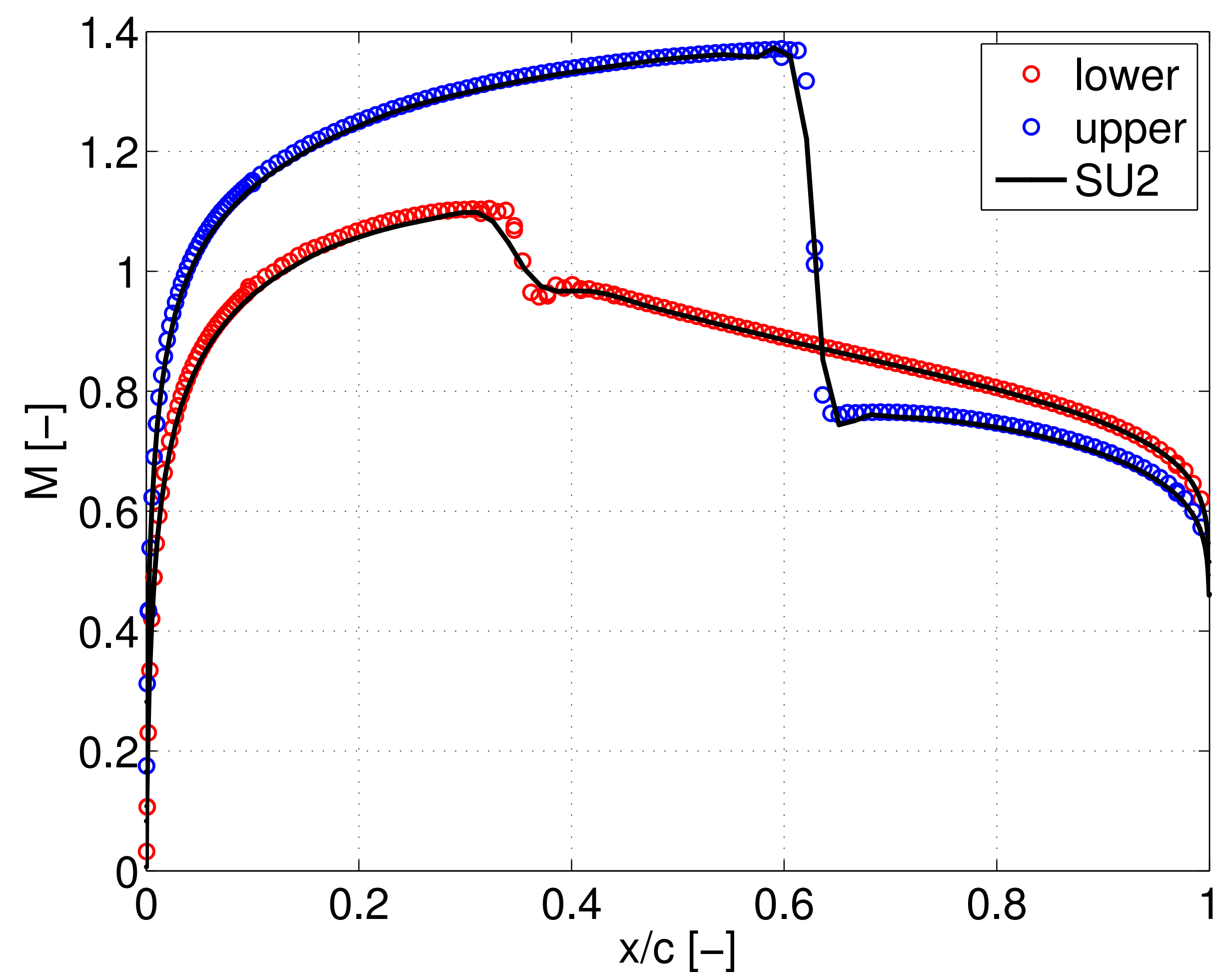
Solver for simulating compressible problems:

- explicit DG/FR
- Euler/Navier-Stokes
- parallel
- shock capturing
- variable-*p* support
- Adjoint (very new)



Shock capturing

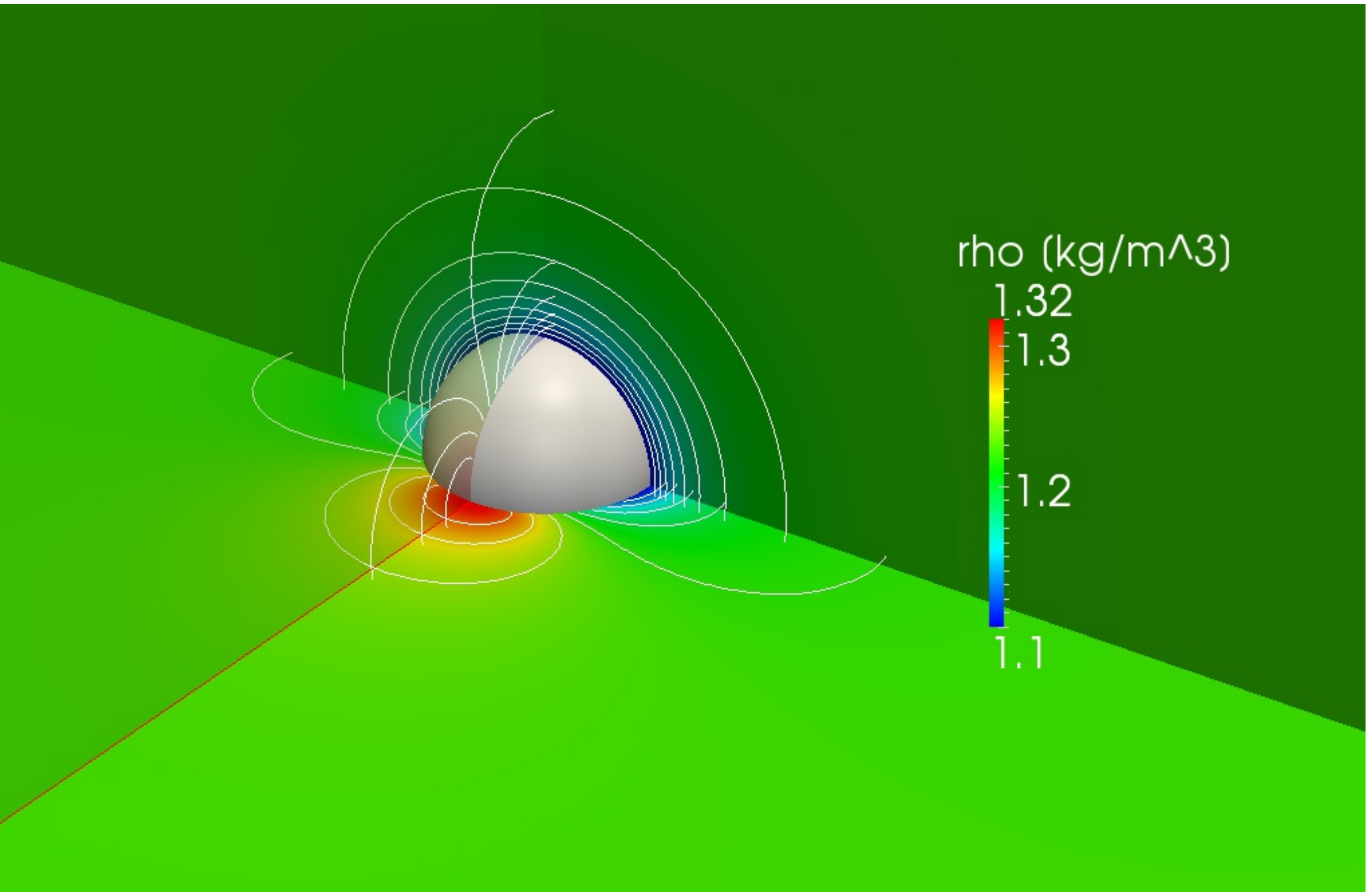




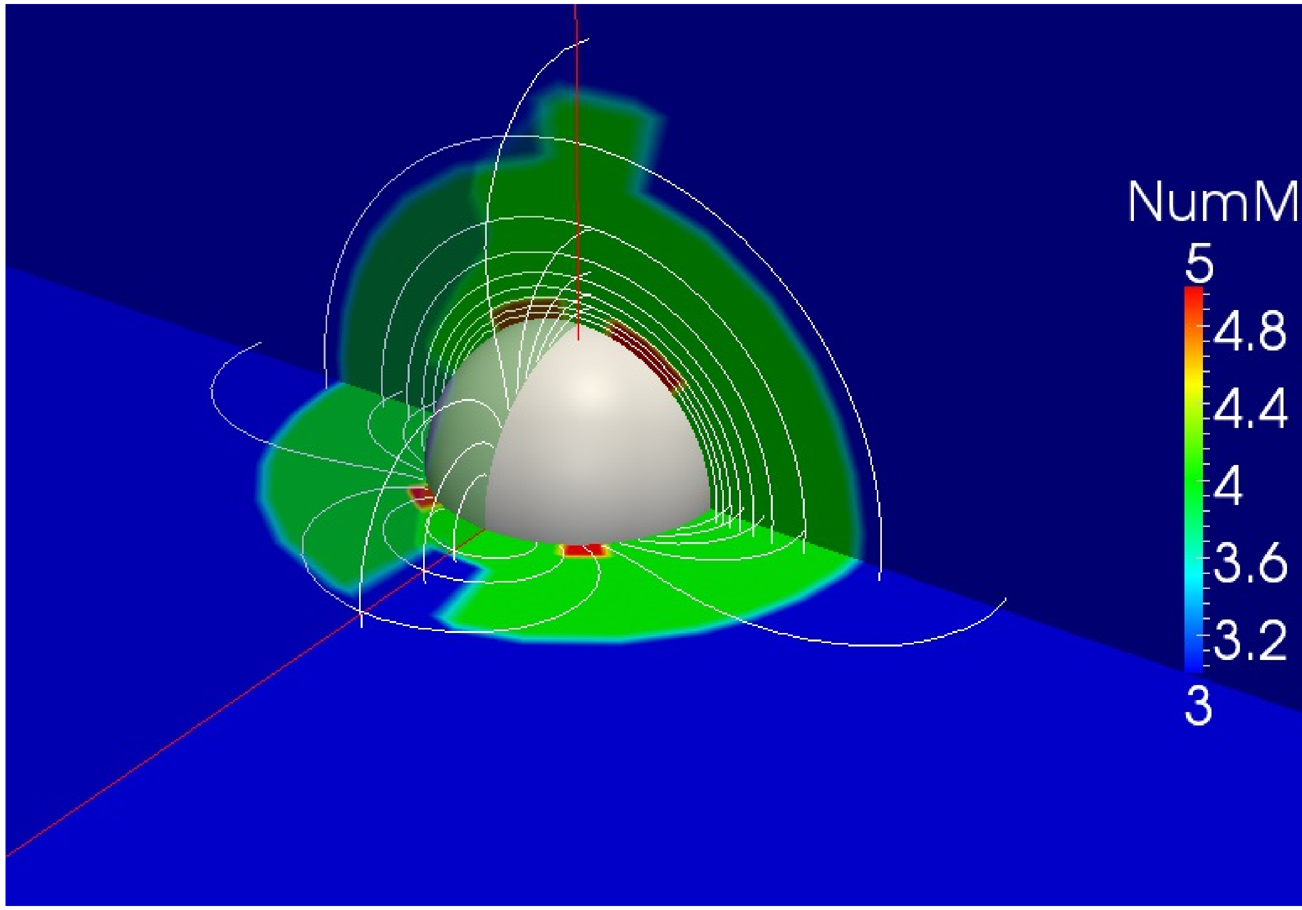
Euler simulation of NACA0012 profile
Sensor + LDG diffusion

Variable-*p*

Euler flow over a sphere



Density



Mode distribution

External aerodynamics
NACA0012 case challenges

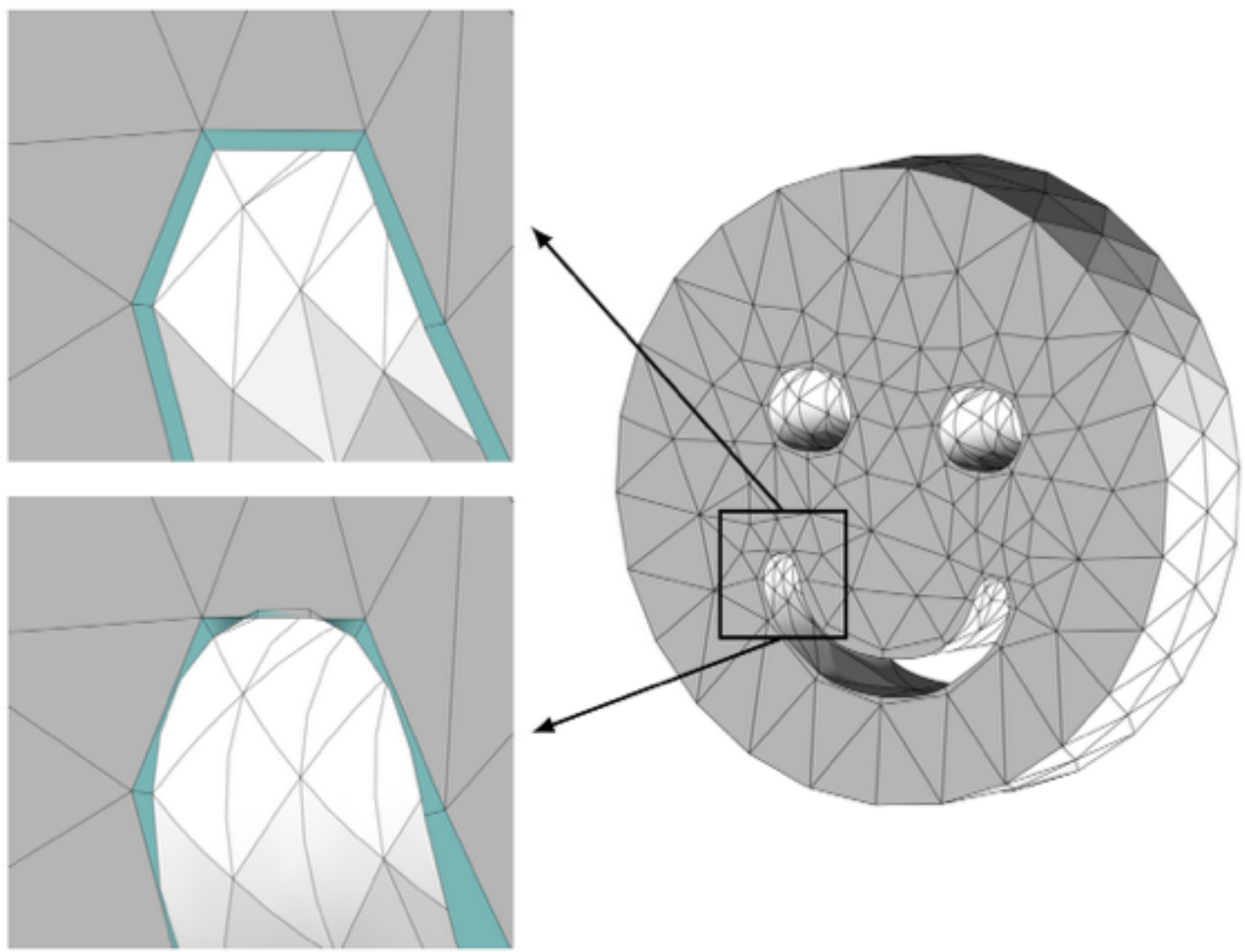
- Stabilisation at high Reynolds number
- Mesh generation

Focus on mesh generation:

- No self-intersecting elements
- Curvilinear elements aligning with geometry
- Deal with boundary layers

High-order mesh generation

Curving mesh often leads to invalid elements



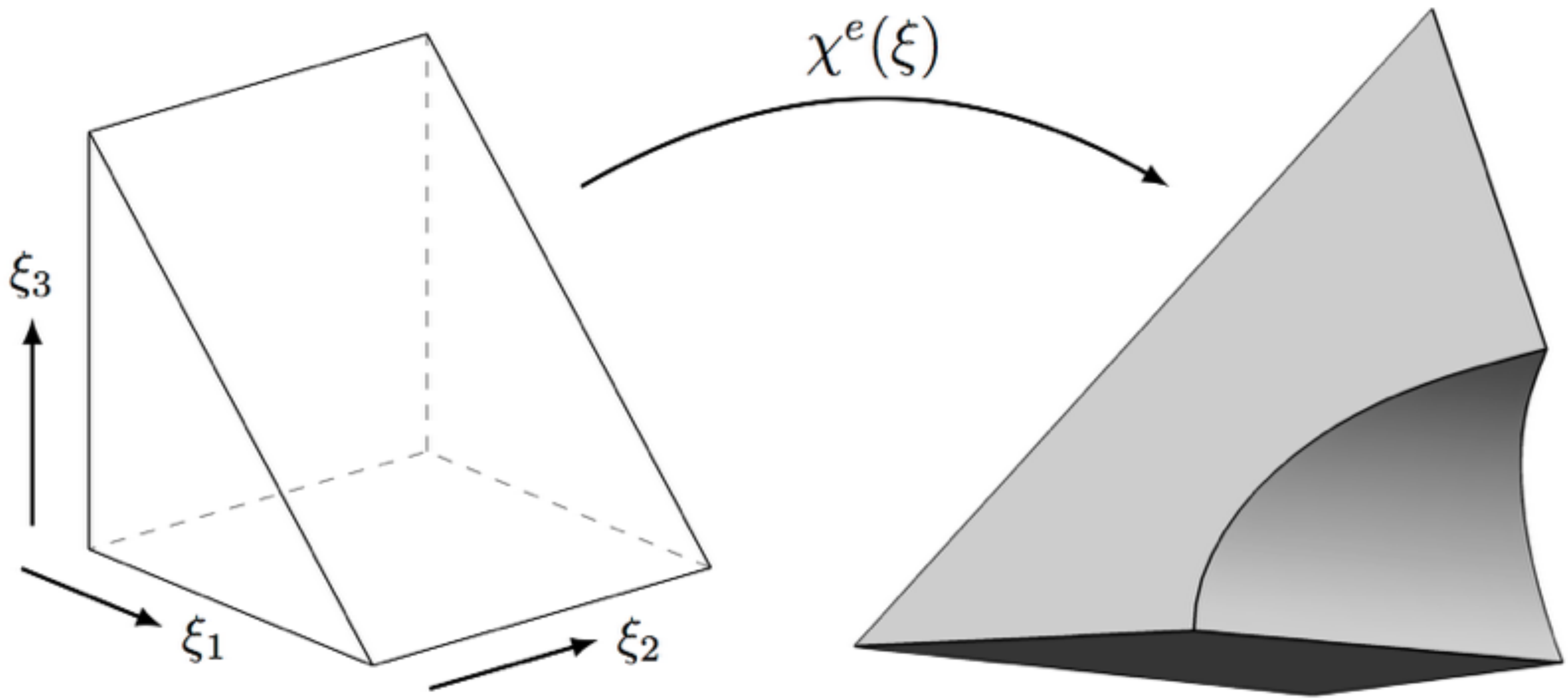
High-order mesh generation

Boundary layer grids are pretty hard to generate:

- High shear near walls
- First element needs to be of size roughly $O(Re^{-2})$
- Unfeasible to run with this number of elements in the entire domain and across surface of wall
- Therefore highly-stretched elements required
- Also has to be coarse for high-order to make sense

What if we already have a coarse grid?

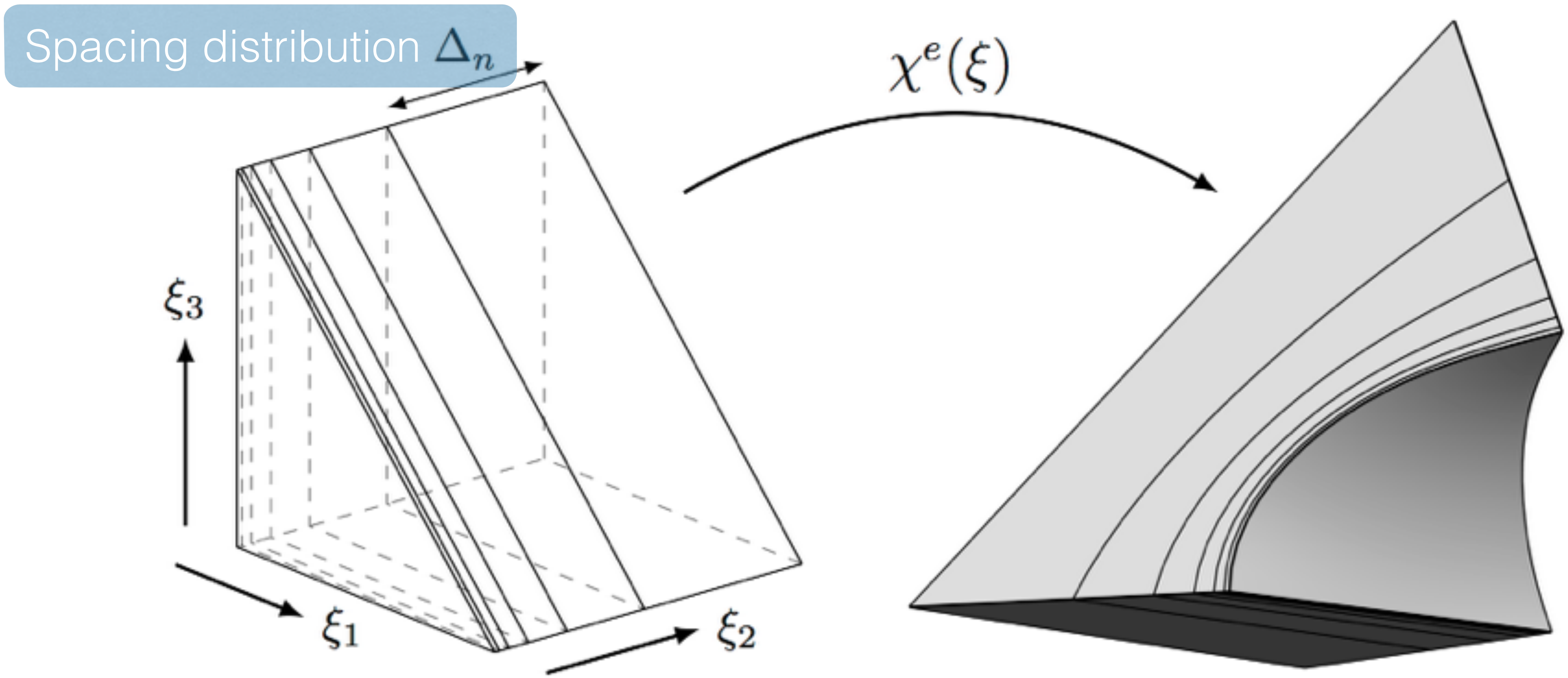
Isoparametric mapping



Shape function is a mapping from reference element (parametric coordinates) to mesh element (physical coordinates)

An isoparametric approach to high-order curvilinear boundary-layer meshing
D. Moxey, M. Hazan, S. J. Sherwin, J. Peiró, under review in Comp. Meth. Appl. Mech. Eng.

Boundary layer mesh generation



Subdivide the reference element in order to obtain a boundary layer mesh

An isoparametric approach to high-order curvilinear boundary-layer meshing
D. Moxey, M. Hazan, S. J. Sherwin, J. Peiró, under review in Comp. Meth. Appl. Mech. Eng.

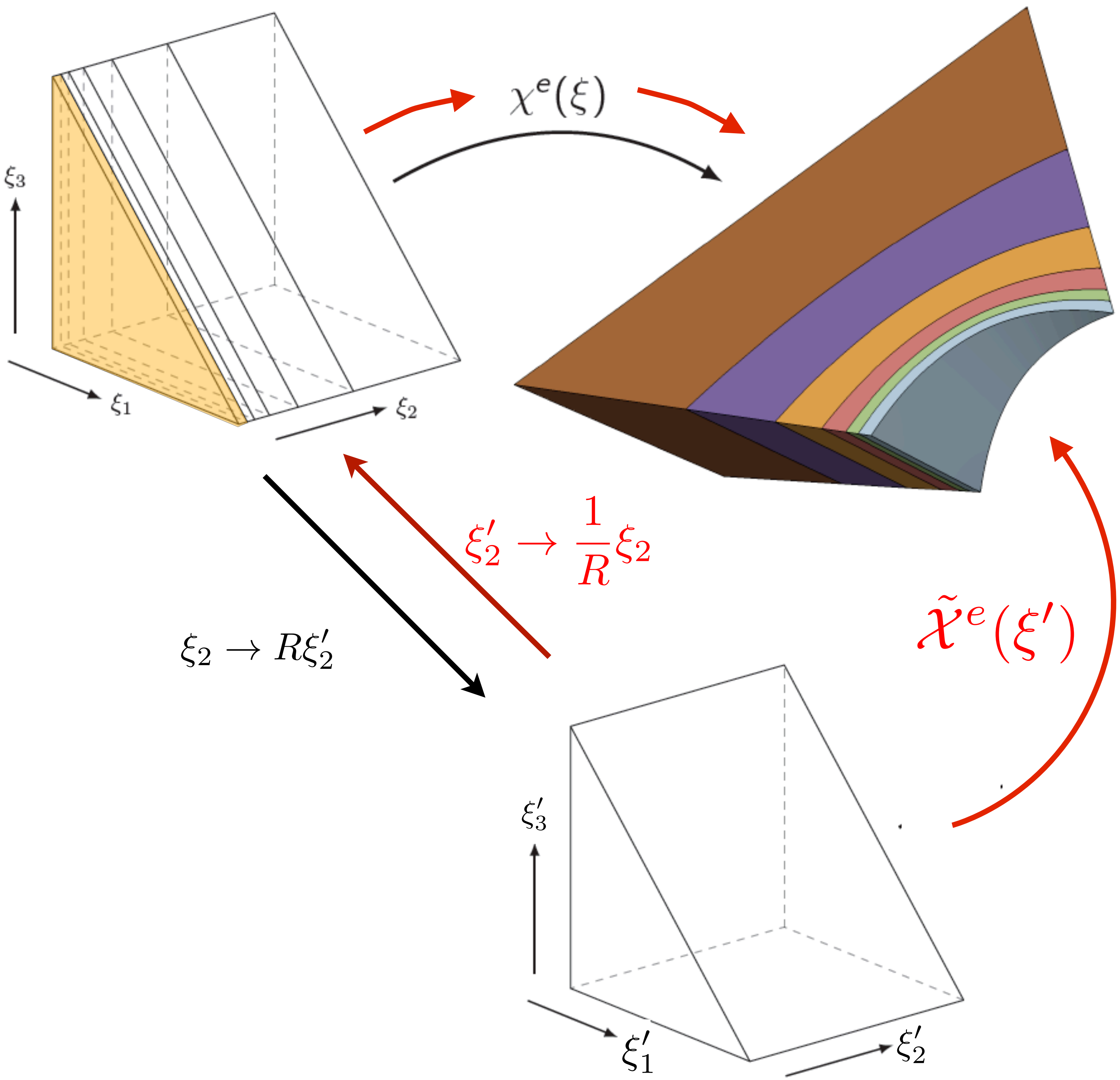
Some nice properties

- Efficient (no deformation required), relatively easy to implement
- Arbitrarily thin elements can be generated near walls (use geometric progression for spacing)
- Guaranteed to produce valid meshes if original mesh is valid thanks to the chain rule
- For same reason, can calculate Jacobian of subelements *a priori*: quality depends on original coarse grid

Some drawbacks

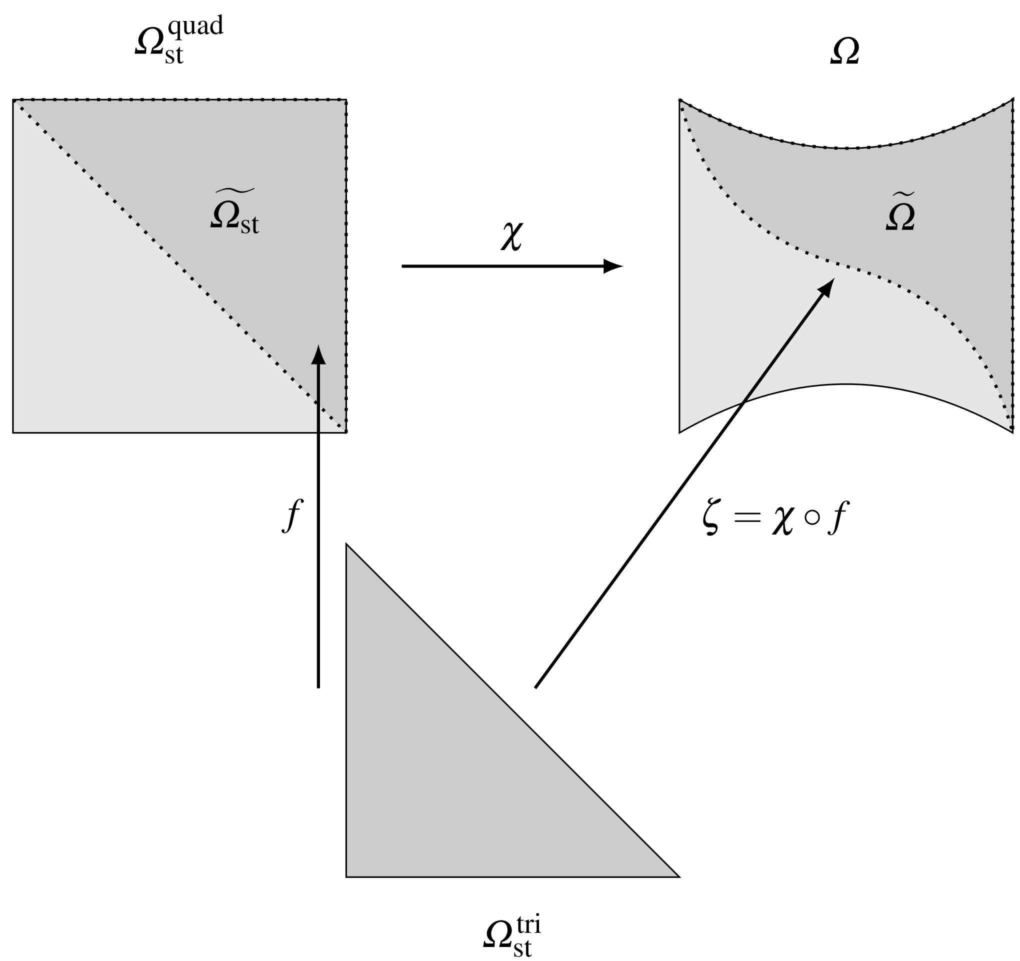
- Relies on O-type geometry unless more complex strategies are undertaken (transition elements)
- Relies on validity and existence of coarse grid
- Mesh quality is dependent on coarse grid

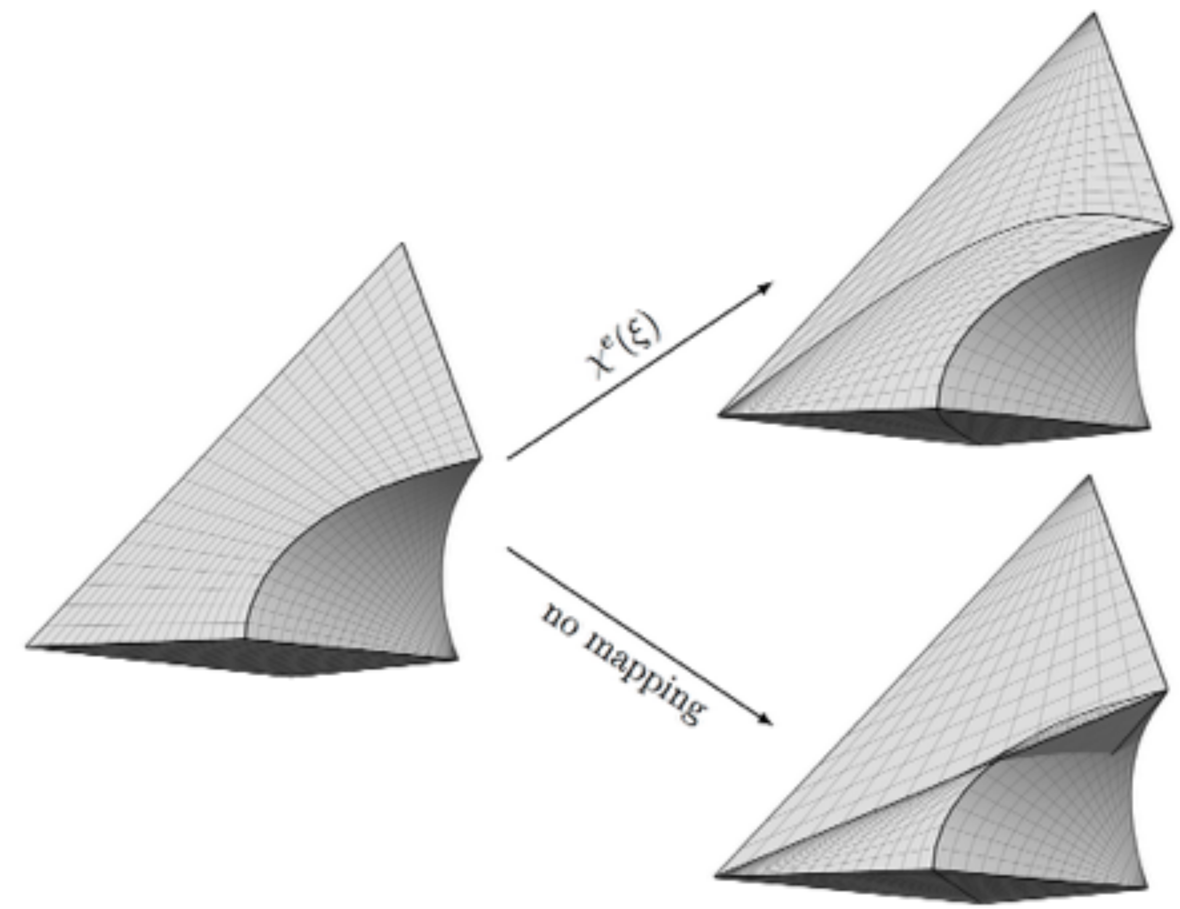
Why does this work?



Jacobian of refined prism is a scaled version of Jacobian of original map

More complex transforms





Quads to triangles

Prisms to tetrahedra

On the generation of curvilinear meshes through subdivision of isoparametric elements
D. Moxey, M. Hazan, S. J. Sherwin, J. Peiró, to appear in proceedings of Tetrahedron IV

Why does this work? (1)

- Write mapping in tensor product of modal functions

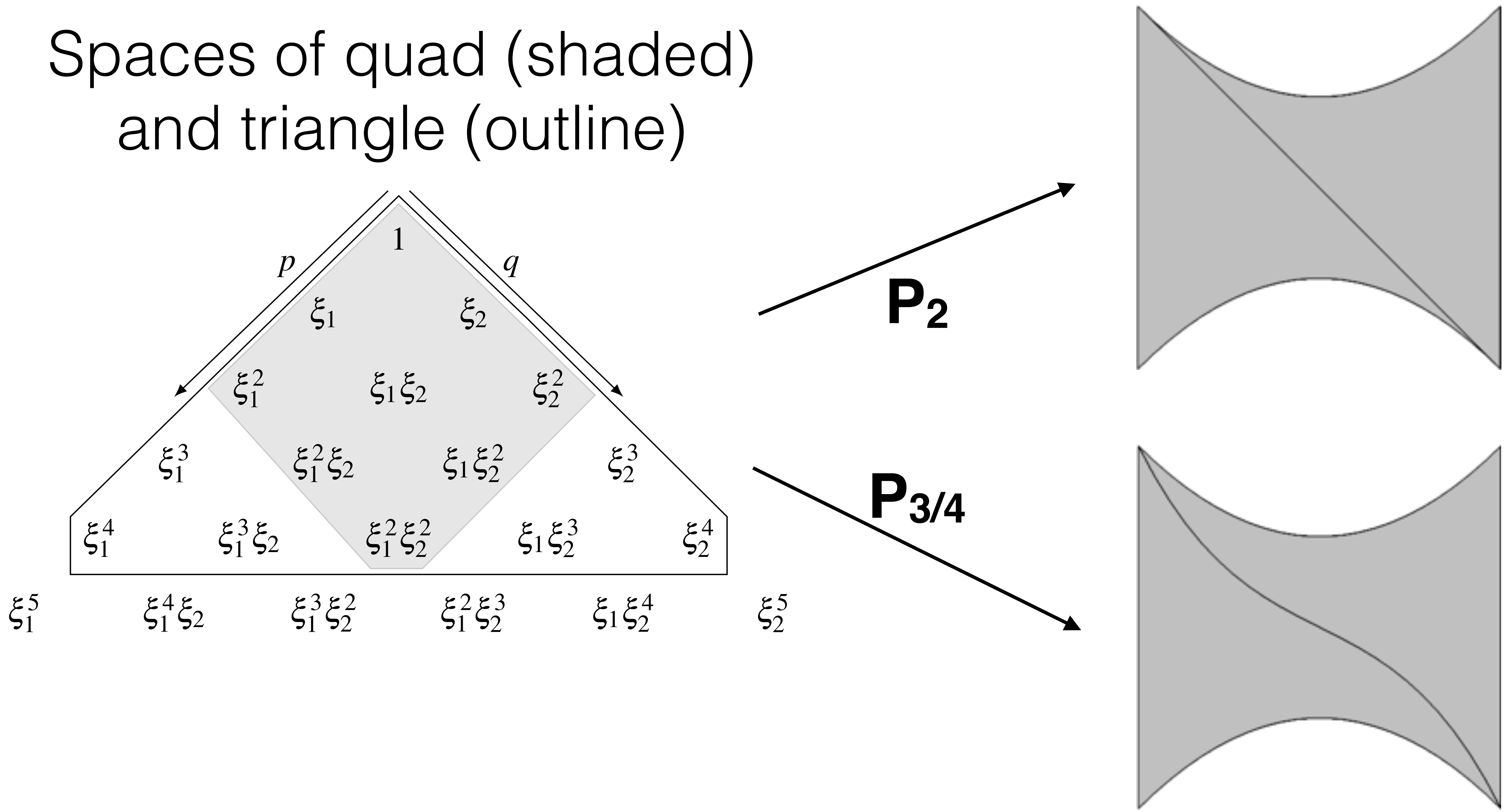
$$\chi_i^e(\xi_1, \xi_2) = \sum_{p,q} (\hat{\chi}_i^e)_{pq} \psi_p(\xi_1) \psi_q(\xi_2)$$

- Then pick polynomial space of target subelement so that it captures all polynomials of original mapping.

- Usually need to enrich subelements to support original mapping but depends on transform.

Why does this work? (2)

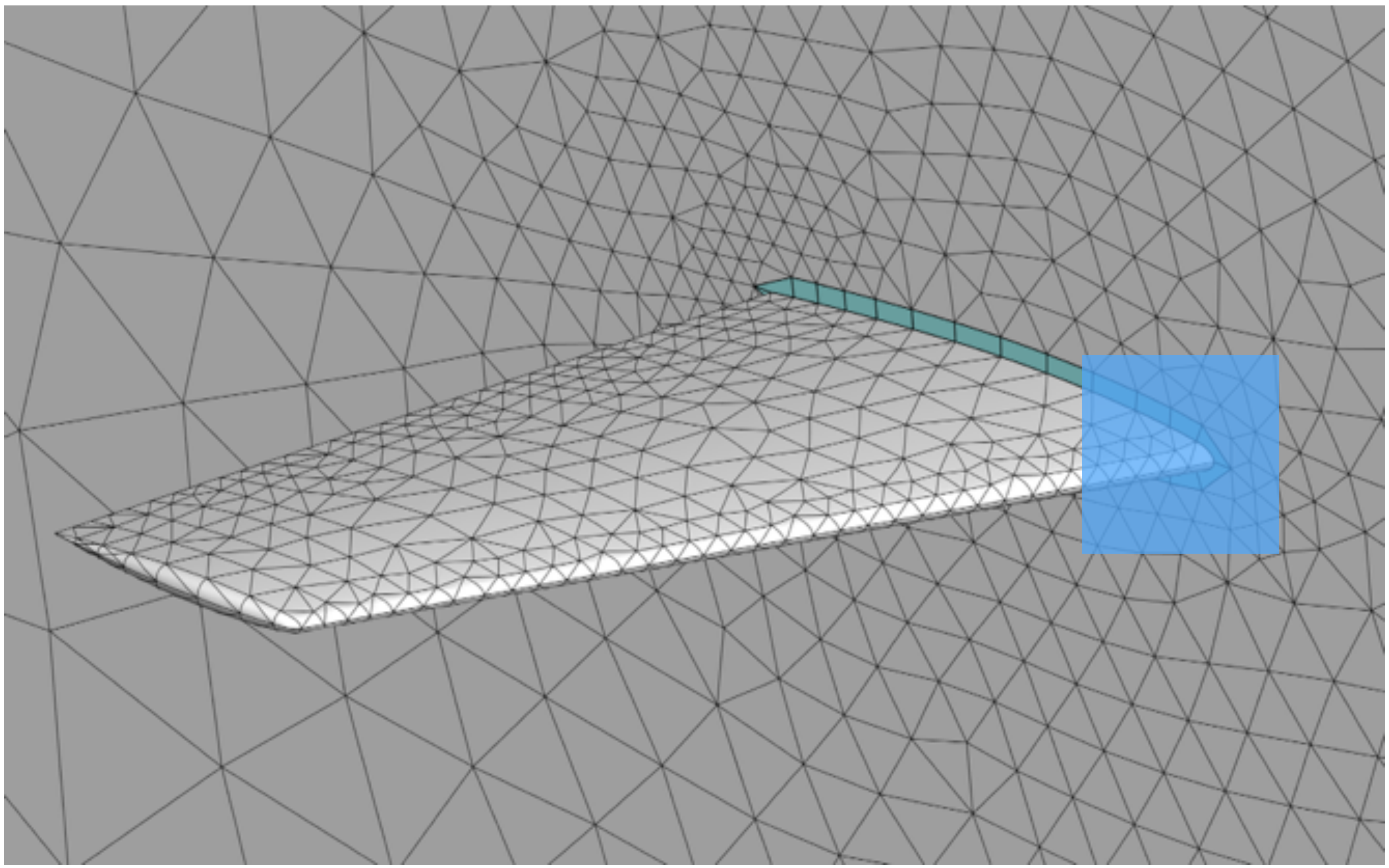
Spaces of quad (shaded) and triangle (outline)



In general: order P quad → order 2P triangle

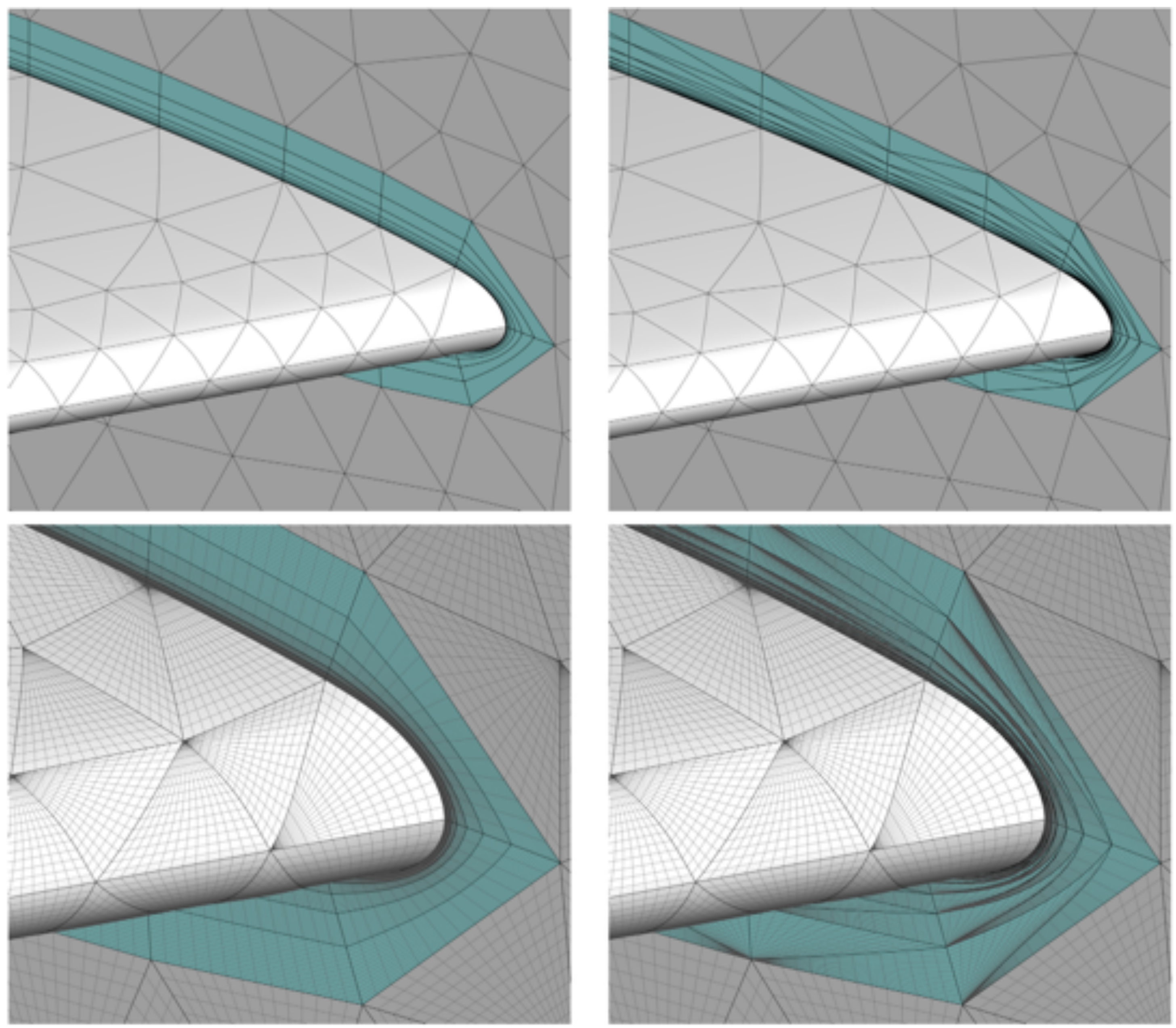
Curvature only in one direction → order P+1 triangle

ONERA M6 wing



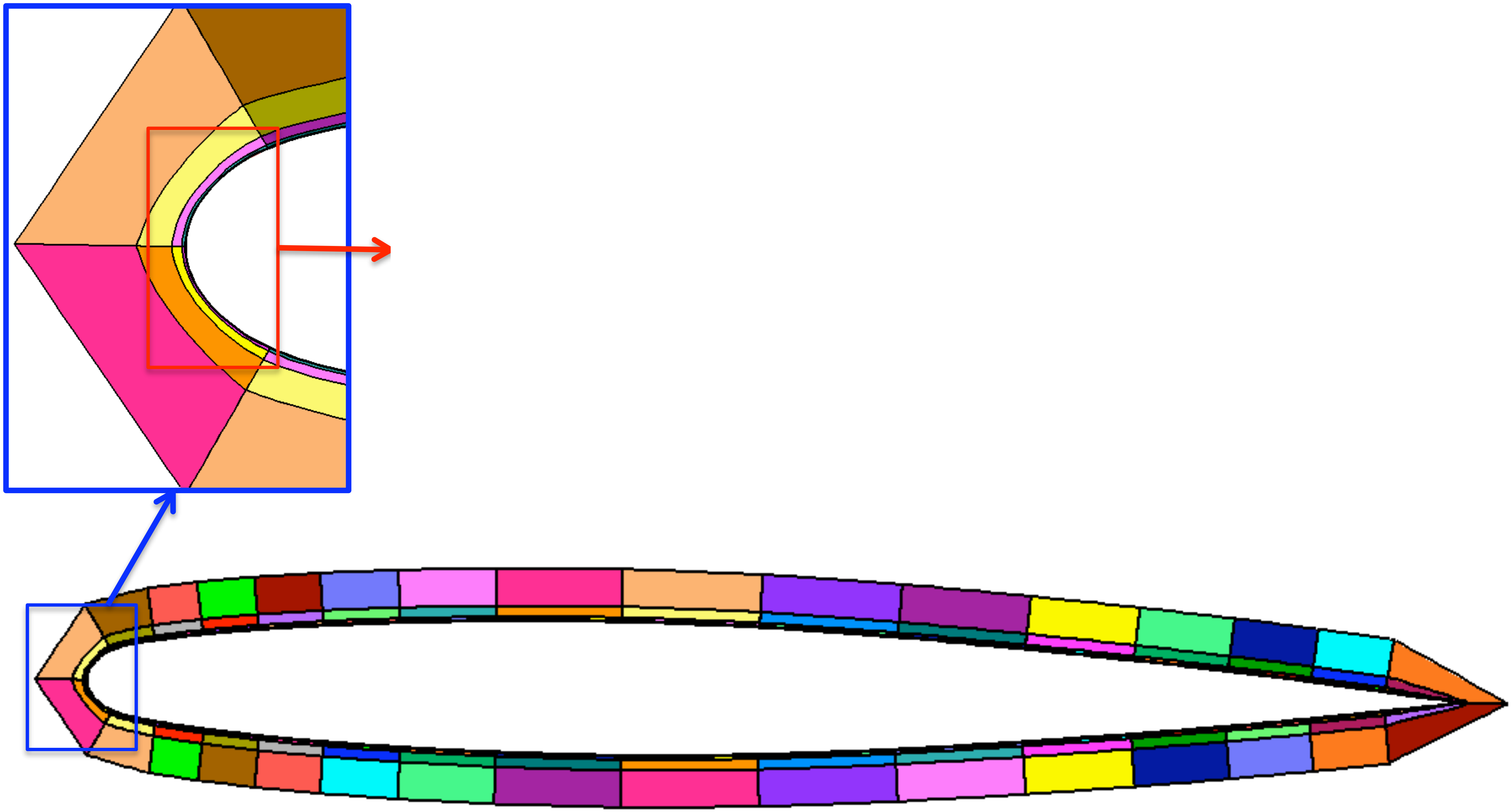
High polynomial order (P = 14)

ONERA M6 wing



Prisms Tets

Proof of concept



7 layers of refinement

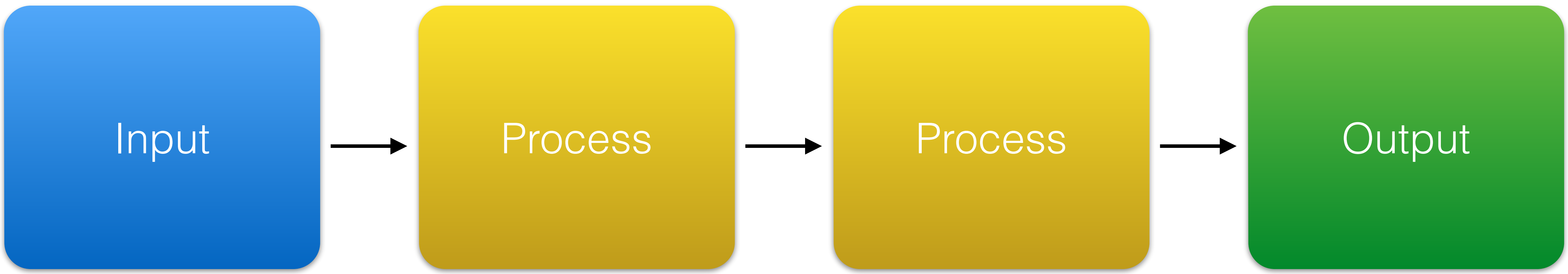
Preprocessing

Many preprocessing techniques:

- Boundary layer refinement
- Simplex element generation
- Surface smoothing
- Surface extraction

Different applications have different requirements: need flexible approach.

Preprocessing



Nektar++
Nektar
Gmsh
Tecplot
StarCCM+
PLY
VTK
Semtex

High-order
smoothing

Boundary
refinement

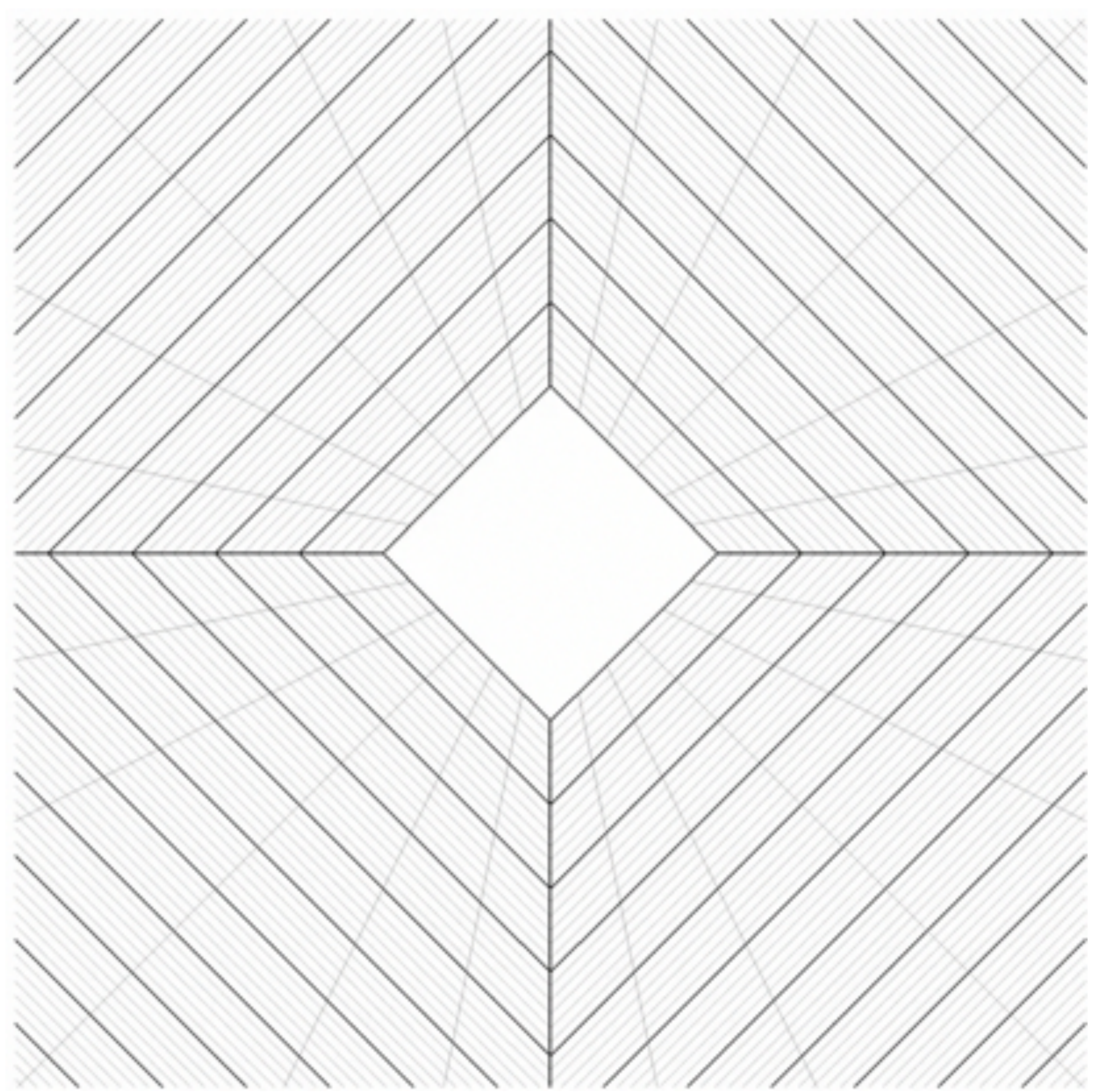
Nektar++
Gmsh
VTK (linear)

MeshConvert: Utilises Nektar++ libraries with pipeline concept: makes preprocessing easier

(in theory)

Current work: deformation

What about coarse grid generation?



Current work: deformation

Utilise linear elasticity equations for displacement **u**

$$\nabla \cdot \mathbf{S} + \mathbf{f} = \mathbf{0} \quad \text{in } \Omega \qquad \mathbf{S} = \lambda \mathrm{Tr}(\mathbf{E})\,\mathbf{I} + \mu \mathbf{E}$$

$$\mathbf{u} = \mathbf{g} \quad \text{in } \partial\Omega \qquad \mathbf{E} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$$

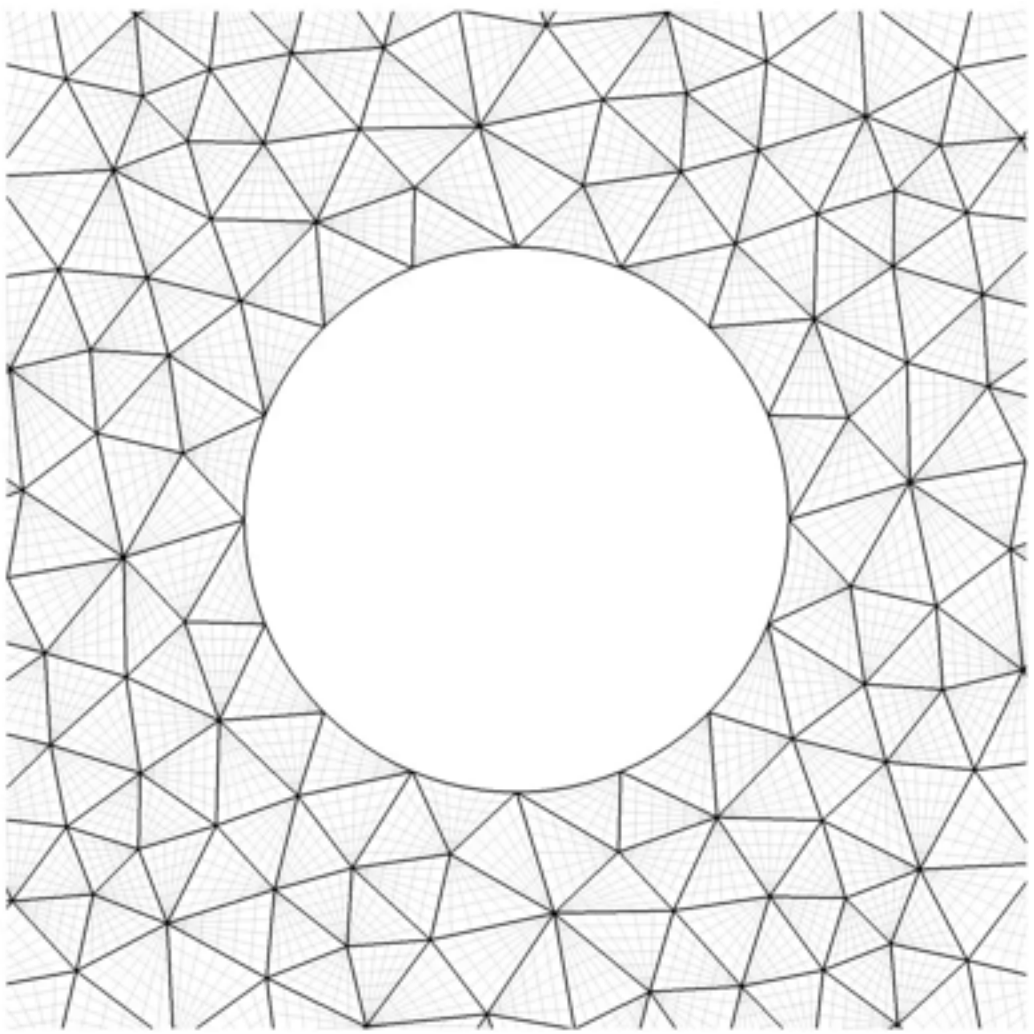
Modification: add thermal stresses to improve robustness so that $\mathbf{S} = \mathbf{S}_e + \mathbf{S}_t$

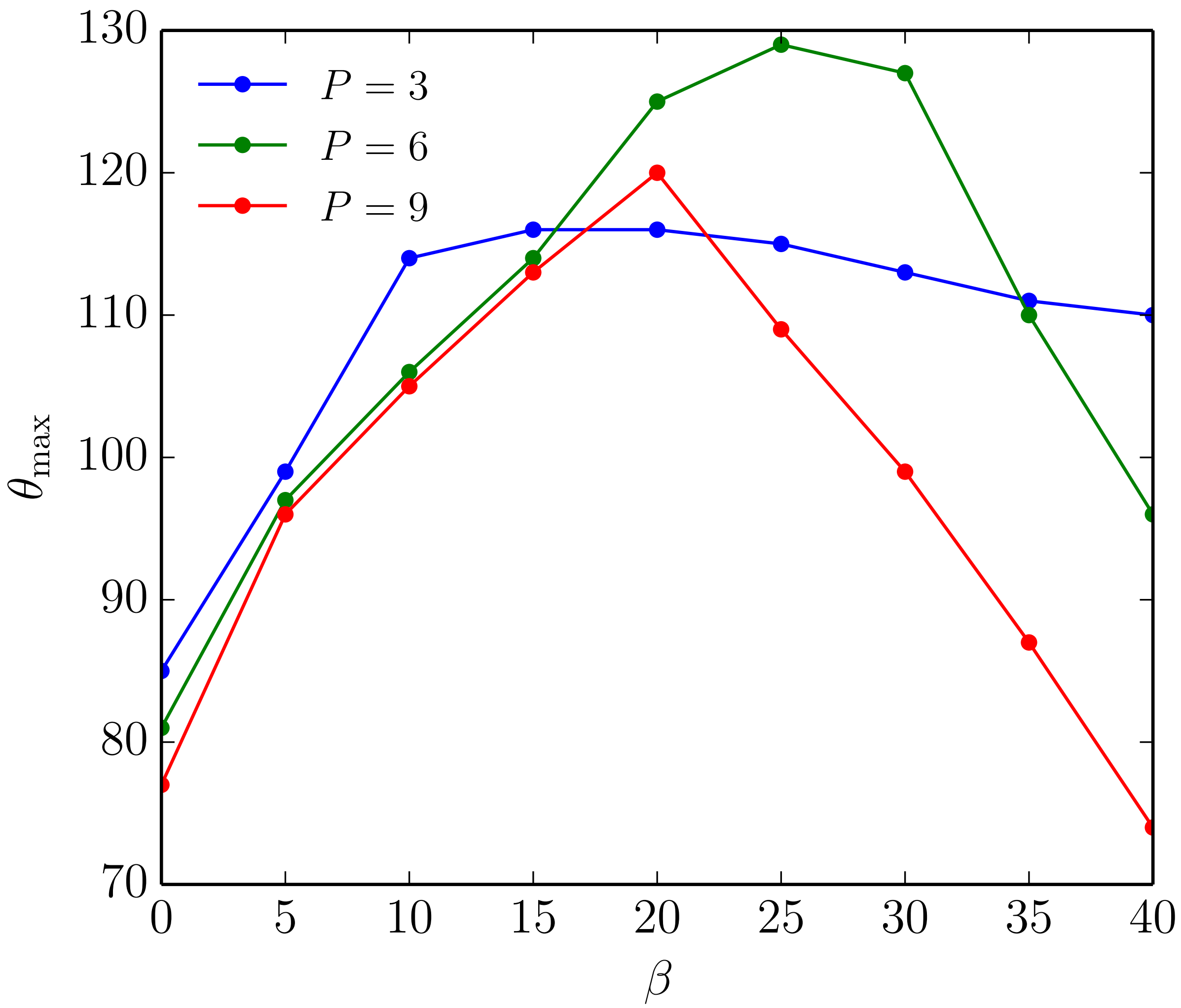
$$\mathbf{S}_t = \beta(T - T_0)\mathbf{I}$$

where T relates to some measure of distortion and T_0 is the temperature of the stress-free state.

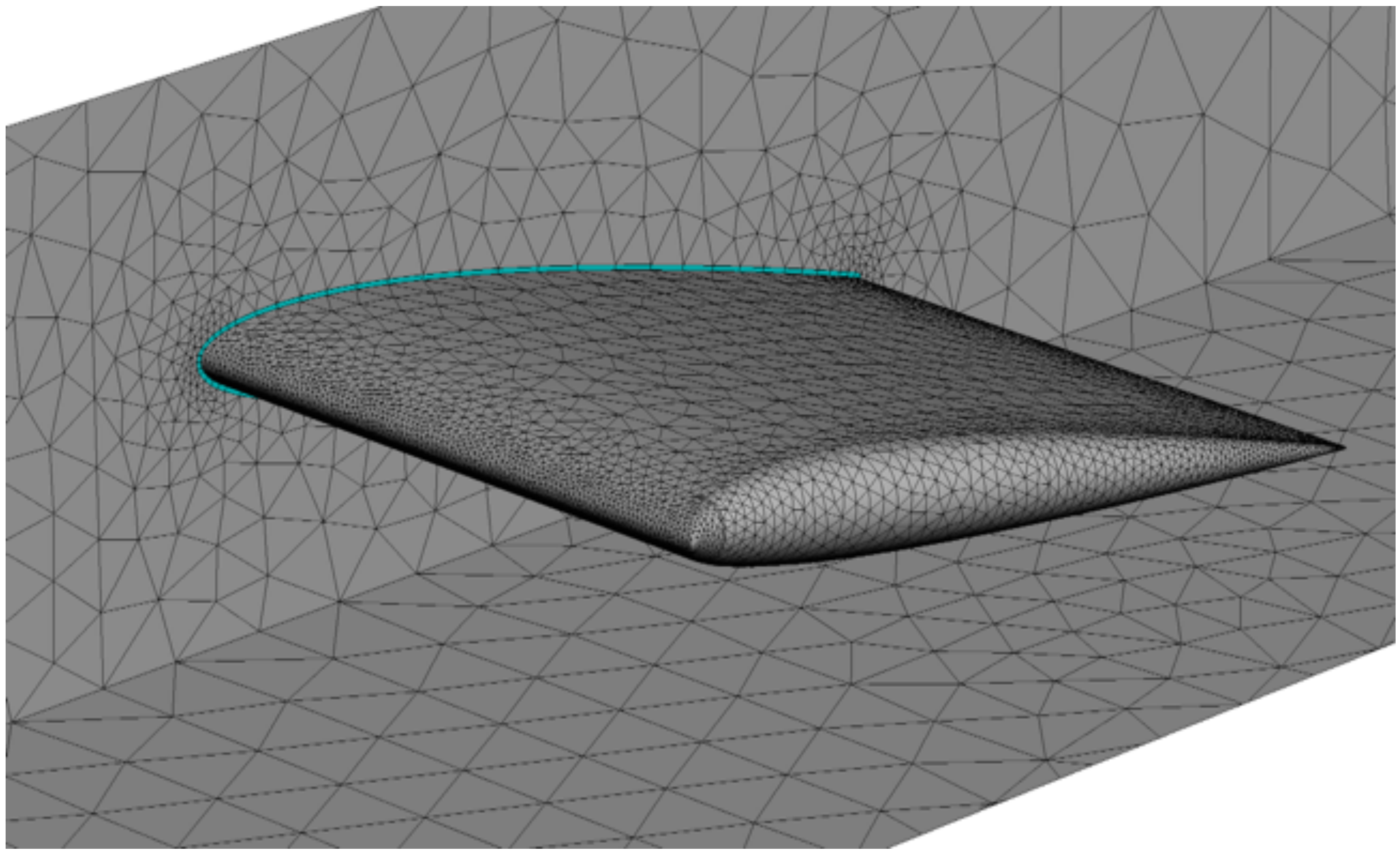
Test case results

- Unstructured triangular mesh of circle inside square
- Rotate circle until occurrence of negative Jacobians



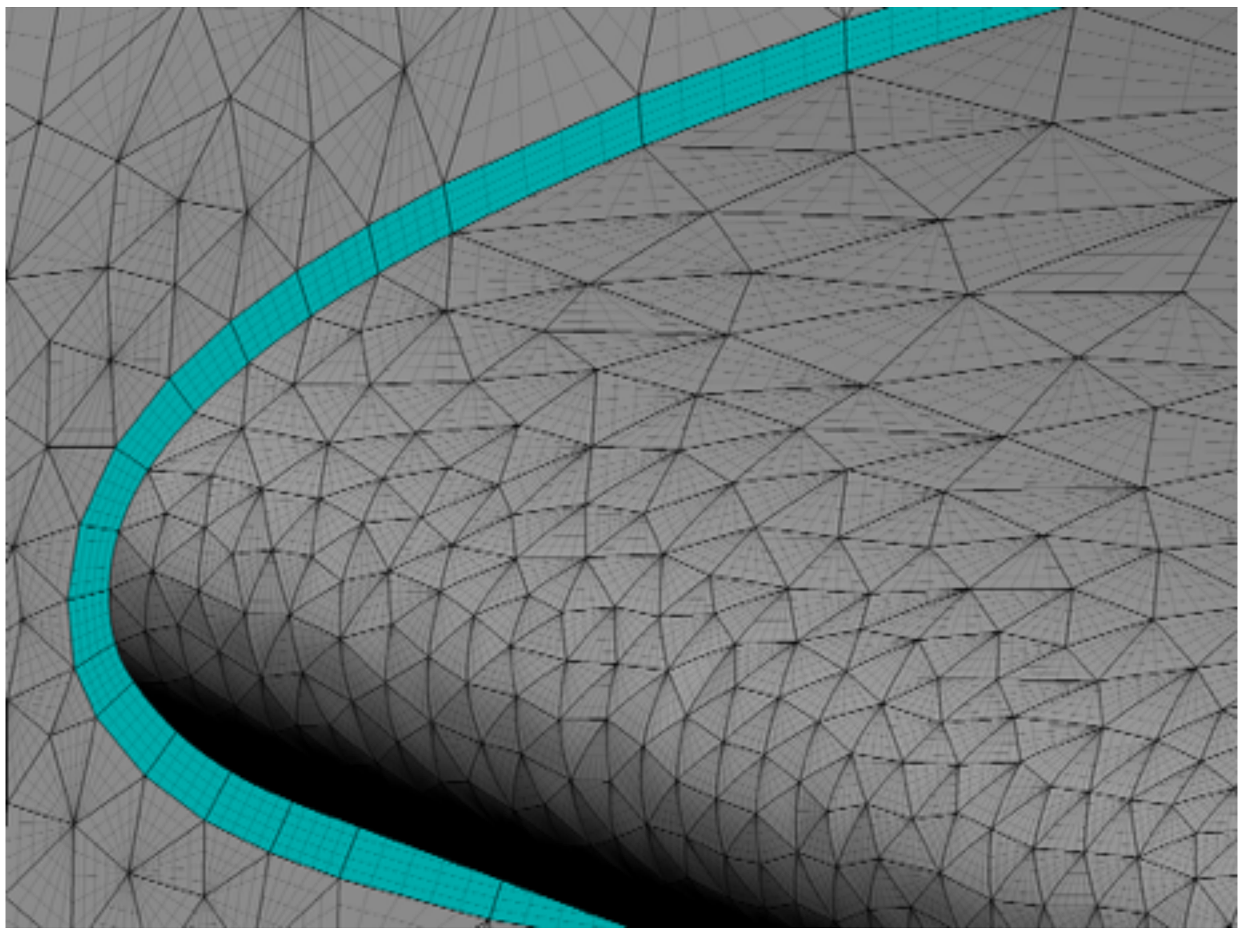


NACA 0012 wing case

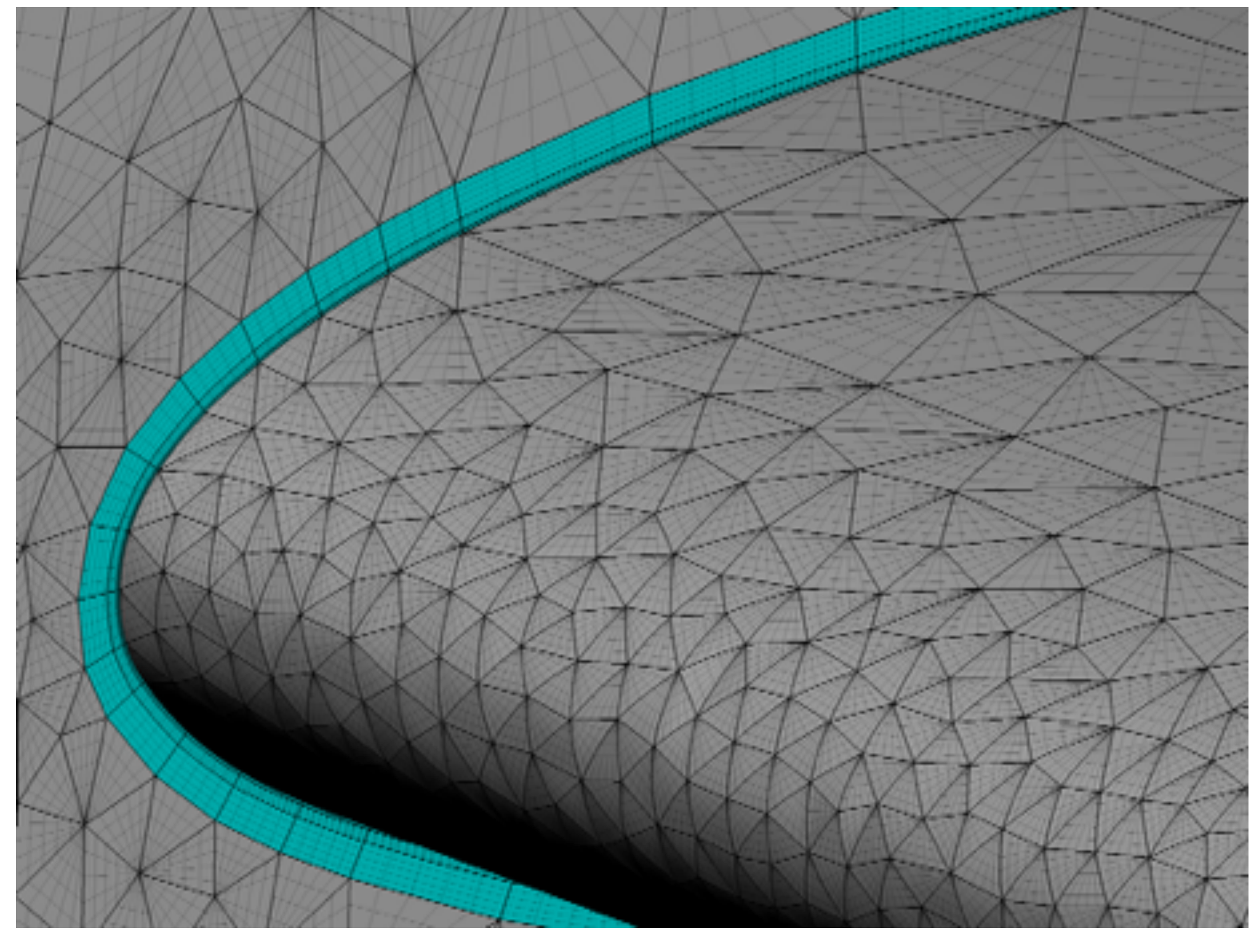


Experimental data at Re ~ 4.5m

NACA 0012 wing case



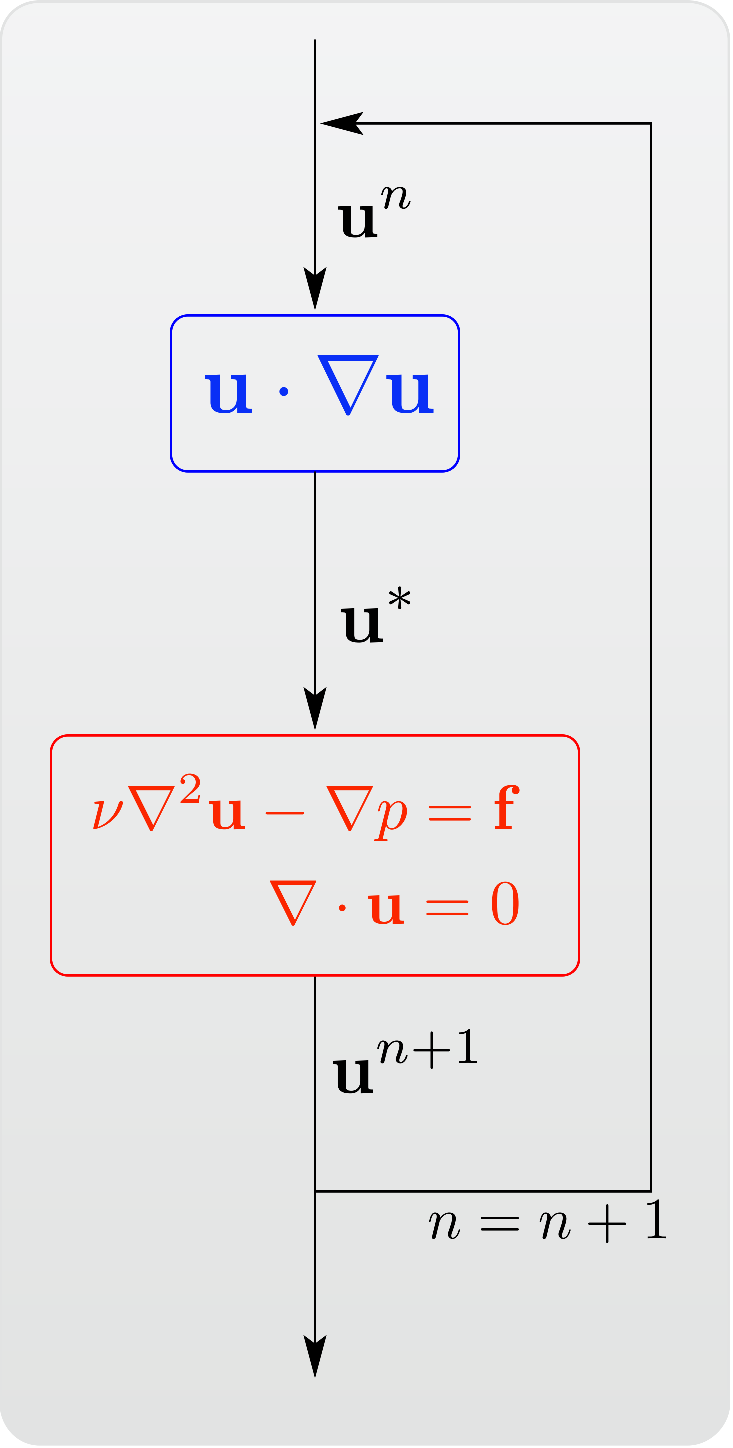
Original high order mesh



Apply splitting technique

Navier-Stokes Solver

Navier–Stokes: $$\partial_t \mathbf{u} + \mathbf{N}(\mathbf{u}) = -\nabla p + \nu \nabla^2 \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0$$



Velocity correction scheme *(aka stiffly stable):*

Orszag, Israeli, Deville (90), Karnaidakis Israeli, Orszag (1991), Guermond & Shen (2003)

Advection: $$u^* = -\sum_{q=1}^{J} \alpha_q \mathbf{u}^{n-q} - \Delta t \sum_{q=0}^{J-1} \beta_q \mathbf{N}(\mathbf{u}^{n-q})$$

Pressure Poisson: $$\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \mathbf{u}^*$$

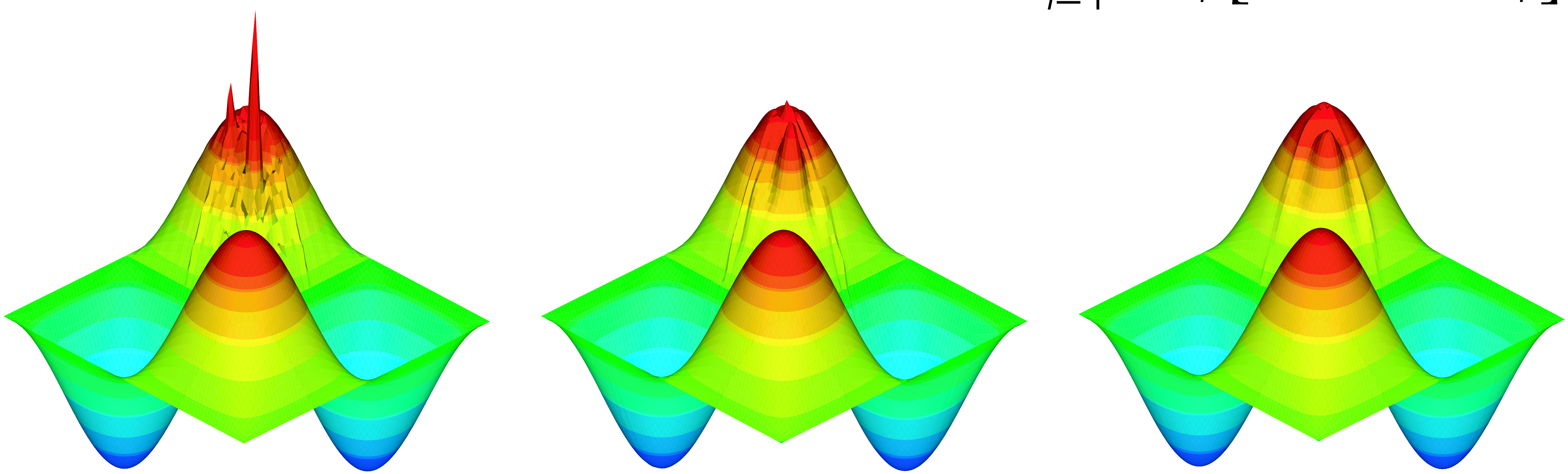
Helmholtz: $$\nabla^2 \mathbf{u}^{n+1} - \frac{\alpha_0}{\nu \Delta t} \mathbf{u}^{n+1} = -\frac{\mathbf{u}^*}{\nu \Delta t} + \frac{1}{\nu} \nabla p^{n+1}$$

Key aspects for simulation

- Stabilisation of numerics: Spectral Vanishing Viscosity is a temporal smoothing/filtering
- Tadmor (89); Maday, Kaber & Tadmor (93).
- Used by Pasquetti, Stiller for high-Re simulation.
- Also require dealiasing/consistent integration of non-linear terms.

Spectral Vanishing Viscosity

$$\frac{\partial u}{\partial t} = v\nabla^2 u + S_{VV}(u), \qquad S_{VV}(u) = \varepsilon \sum_{i=1}^{\dim} \frac{\partial}{\partial x_i}\left[Q_{\dim} \star \frac{\partial u}{\partial x_i} \right]$$



No SVV

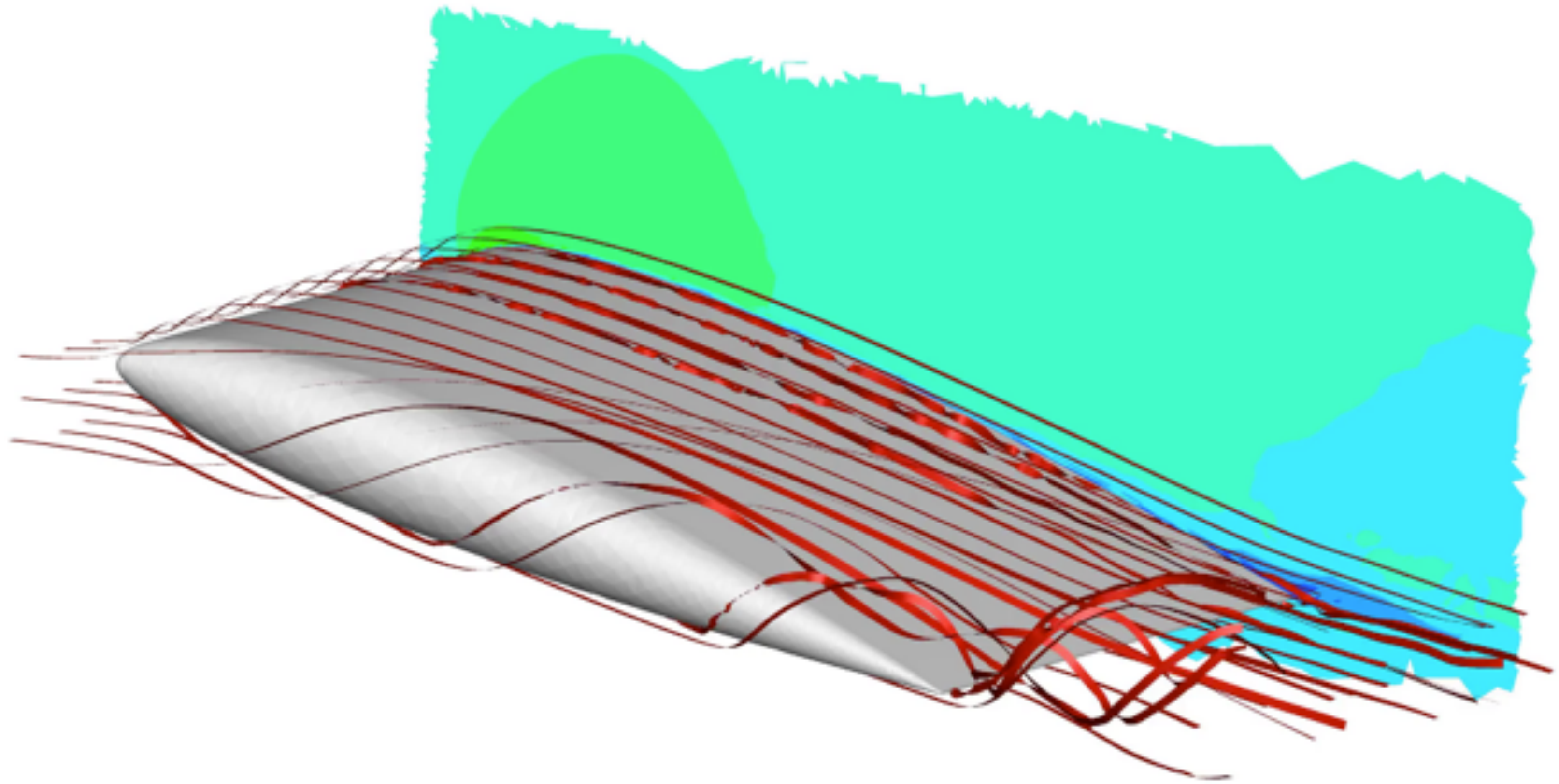
$P_{cut} = 7,$
$\epsilon_{SVV} = 0.1$

$P_{cut} = 3,$
$\epsilon_{SVV} = 0.1$

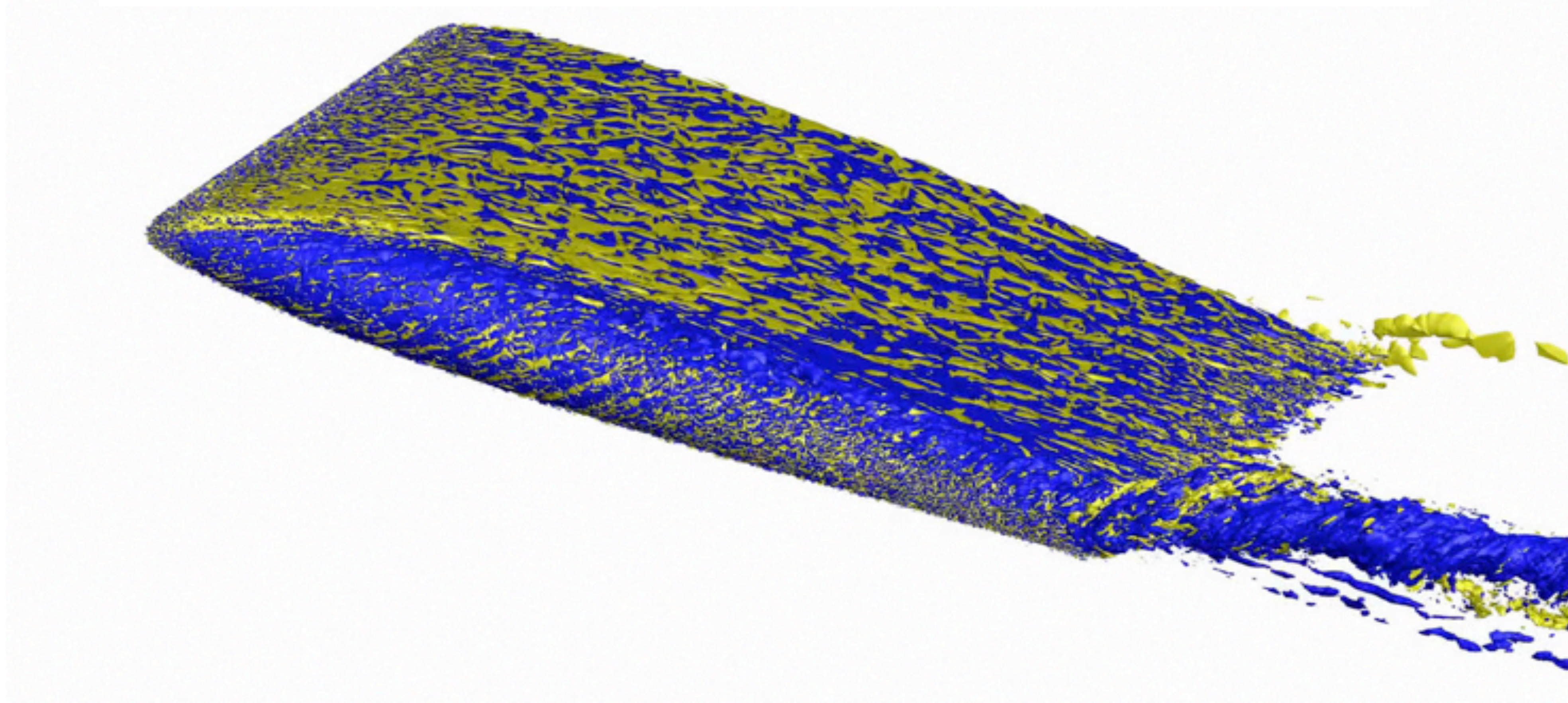
R. Kirby, S. Sherwin, *Comp. Meth. Appl. Mech. Eng.*, 2006

NACA 0012 wing tip (Re = 1.2M)

Streamlines

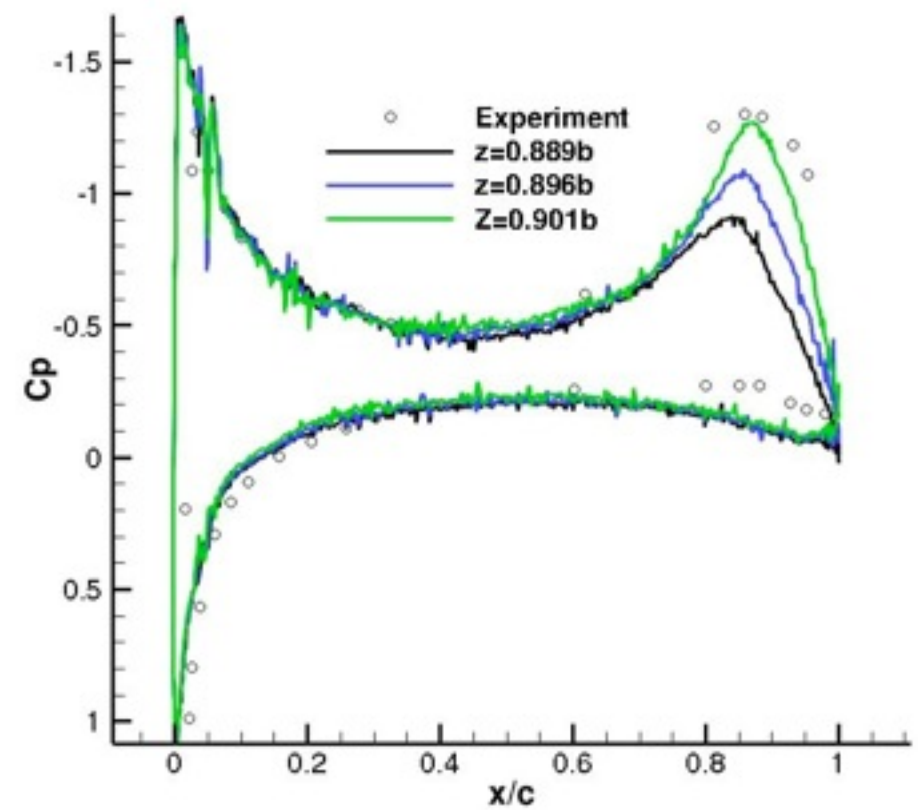


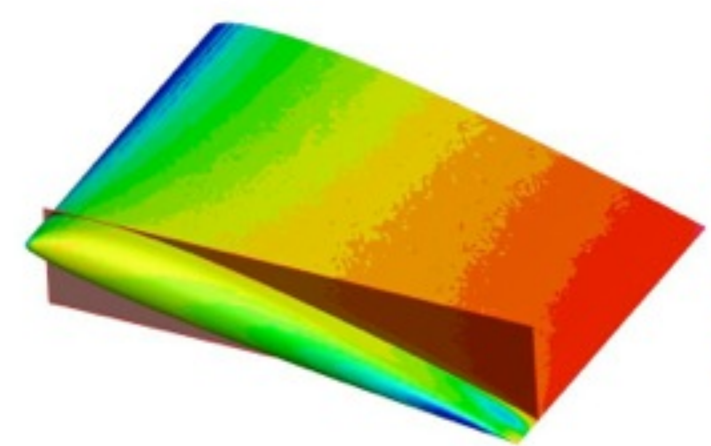
Streamwise vorticity



NACA 0012 comparisons

- DNS stabilised by using spectral vanishing viscosity (SVV) at polynomial order 5
- Wingtip vortex is captured
- Cut-plane of pressure distribution shows good agreement with experiment
- Still need to investigate effects of dealiasing on solution field





Conclusions

- High-order methods can be applied to these problems and successfully capture essential flow dynamics
- Still a need for high-order mesh generation strategies for coarse grid
- Future: grid deformation for coarse grid with emphasis on element quality

http://www.nektar.info/

Thanks for listening!