On the generation of curvilinear meshes through subdivision of isoparametric elements

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Abstract Recently, a new mesh generation technique based on the isoparametric representation of curvilinear elements has been developed in order to address the issue of generating high-order meshes with highly stretched elements. Given a valid coarse mesh comprising of a prismatic boundary layer, this technique uses the shape functions that define the geometries of the elements to produce a series of subdivided elements of arbitrary height. The purpose of this article is to investigate the range of conditions under which the resulting meshes are valid, and additionally to consider the application of this method to different element types. We consider the subdivision strategies that can be achieved with this technique and apply it to the generation of meshes suitable for boundary-layer fluid problems.

1 Introduction

In recent years, interest in high-order finite element methods has increased dramatically. Their attractive dispersion properties, exponential convergence of approximate solutions and computational performance when compared to traditional low-order methods make high-order methods an attractive prospect for researchers in both academia and industry across a wide range of application areas. However, one of the main issues to be overcome before these methods can be widely adopted is the

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development of methods for reliable generation of curvilinear meshes for complex three-dimensional domains.

A particularly important problem to be solved is the generation of meshes where highly stretched elements are desired, either as function of the geometry or for reasons of computational cost. In a typical high-order generation process one begins by generating a linear mesh, and then deforming the elements connected to curved surfaces by projecting points lying on the surface geometry into the interior of each face. When the linear elements are highly stretched so that their thickness is small, then in regionswhere the curvature of the geometry is high, deforming only the elements connected to the surface can result in self-intersection and thus the mesh becomes unsuitable for computations [5]. Whilst techniques exist to deform linearly refined meshes using linear or non-linear elastic analogies [7, 4] or alternatively untangle and optimise meshes which have self-intersection [6], they are relatively expensive and their success has been limited when applied to these problems.

One solution to this problem that has been recently proposed by the authors of this work is to consider an isoparametric approach to producing highly stretched meshes [3]. Given an existing valid high-order mesh of prismatic elements, in this technique one subdivides each prism by using the shape functions defining the geometry of the original prismatic element. This method is simple, cheap to implement and leads to the generation of meshes which are guaranteed to be valid so long as the original mapping is valid. Furthermore, this subdivision technique can be adapted to address other issues, such as the generation of meshes containing only high-order simplex elements for solvers which do not support hybrid meshes.

The purpose of this paper is to frame the subdivision technique in the context of a more general mathematical framework and demonstrate how it can be utilised to subdivide a broader range of elemental types in both two and three dimensions. We note that in general, the subdivision of elements in this manner often requires the enrichment of the polynomial space so that the subdivided elements capture all curvature of the original element. One of aims of this paper therefore is to establish the necessary conditions for the validity of the resulting subelements under various subdivision strategies, and demonstrate through numerical examples the applicability of the method to mesh generation and that such conditions are indeed required.

The paper is structured as follows. Section 2 outlines the motivation for the subdivision technique and gives a brief overview of the process through which an element is subdivided as presented in [3]. The mathematical framework for a generalisation of the method to other element types is given in section 3. We then demonstrate some applications of the method in section 4 to problems in aeronautics and biomechanics, and the subdivision of elements to produce meshes containing only simplex elements. Finally we conclude with some remarks on further applications and improvements in section 5.

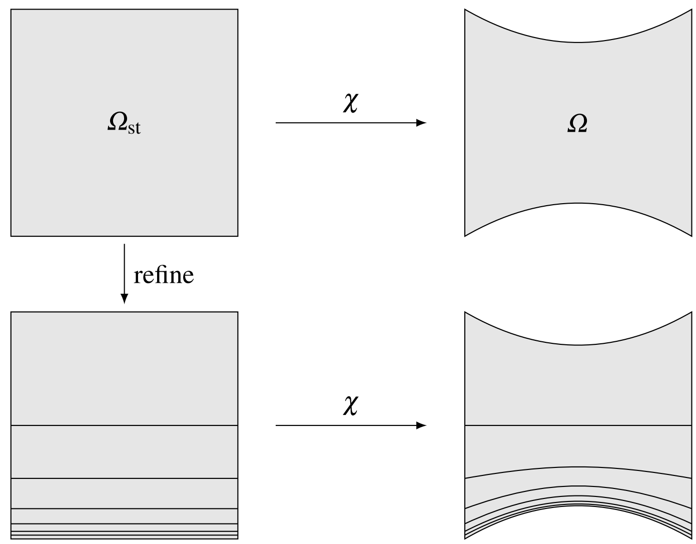

Fig. 1 Overview of boundary layer refinement technique presented in [3].

2 Motivation

One of the main application areas of high-order methods is the simulation of fluid flow over aeronautical geometries where, near walls, flow gradients in the direction normal to the wall are several orders of magnitude larger than those tangential to the wall. Sufficient resolution in the near-wall boundary layer in order to resolve this shear is crucial since the vortices which lead to turbulent instabilities develop close to this region, and so under-resolution will usually lead to unphysical or biased results. In these simulations therefore, the size of elements in wall-normal directions must be small so that the steep gradient of the near-wall flow profile is adequately resolved. In the other directions however, such resolution is not usually required, leading to the generation of elements with a large stretching ratio. Introducing curvature into elements near the boundary layer will often lead to self-intersection in regions of high curvature.

The isoparametric subdivision technique proposed in [3] addresses this problem. Firstly, we assume that a coarse mesh, comprising of a prismatic boundary layer and tetrahedra elsewhere, has been generated. As part of the usual high-order mesh generation procedure, we construct a mapping χ which maps coordinates ξ in a reference element into the Cartesian coordinates of Ω. In order to produce a series of refined prismatic elements in the physical domain, we instead refine the standard elemental region, and utilise the mapping χ to map this back into physical space.

An overview of this process can be seen for a representative quadrilateral element in figure 1, where we assume the bottom edge of the element is attached to the wall, and therefore require extra resolution in the vertical direction. The top row shows how the standard quadrilateral element $\Omega_{\rm st}$ is deformed under the mapping χ to produce a curved element Ω. To refine the element in physical space, we first split $\Omega_{\rm st}$ into a series of smaller elements as shown in bottom left of the figure. Applying the mapping χ to these subelements of the standard region leads to the production of curved subelements of the physical element as desired.

What remains to be presented, and indeed is the focus of the rest of this paper, is an examination of the conditions under which the subdivision process produces valid elements, not only for the refinement procedure outlined here, but for more generic transformations of the standard element. In the following section we describe the mathematical framework in which this problem is defined, and describe more precisely how a subdivision procedure impacts the polynomial spaces which define the elements.

3 Mathematical framework

We begin by providing a brief mathematical framework for the method. We begin by considering a finite element Ω^e, which in general belongs to a mesh arising from the tesselation $\mathscr{T}(\Theta) = \{\Omega^1, \dots, \Omega^{N_{\rm el}}\}$ of some domain $\Theta \subset \mathbb{R}^n$ with $n = 2, 3$ of $N_{\rm el}$ elements, so that

$$\Theta = \bigcup_{e=1}^{N_{\rm el}} \Omega^e, \qquad \Omega^e \cap \Omega^f = \emptyset \text{ if } e \neq f.$$

In two dimensions, we consider quadrilateral and triangular shaped elements, and in three dimensions tetrahedral, prismatic and hexahedral elements. In order to introduce curvature into an element Ω (where we drop the superscript e for convenience), we assume there exists a mapping $\chi : \Omega_{\rm st} \to \Omega$ which projects a canonical standard element $\Omega_{\rm st}$ into the Cartesian coordinates defining an element. In this work we define reference elements to be

$$\begin{aligned}
\Omega_{\rm st}^{\rm quad} &= \{(\xi_1, \xi_2) \quad\ \mid -1 \leq \xi_1, \xi_2 \leq 1\}, \\
\Omega_{\rm st}^{\rm tri} &= \{(\xi_1, \xi_2) \quad\ \mid -1 \leq \xi_1 + \xi_2 \leq 1\}, \\
\Omega_{\rm st}^{\rm hex} &= \{(\xi_1, \xi_2, \xi_3) \mid -1 \leq \xi_1, \xi_2, \xi_3 \leq 1\}, \\
\Omega_{\rm st}^{\rm pri} &= \{(\xi_1, \xi_2, \xi_3) \mid -1 \leq \xi_1 + \xi_3 \leq 1, -1 \leq \xi_2 \leq 1\}, \\
\Omega_{\rm st}^{\rm tet} &= \{(\xi_1, \xi_2, \xi_3) \mid -1 \leq \xi_1 + \xi_2 + \xi_3 \leq 1\},
\end{aligned}$$

respectively. Inside the standard elements we define a polynomial space in terms of the reference coordinates $\xi = (\xi_1, \xi_2, \xi_3)$ from which an expansion basis is selected. Assuming that we select a polynomial order P, Q and R for each coordinate direction, the polynomial spaces take the form

$$\mathscr{P}(\Omega_{\mathrm{st}}) = \mathrm{span}\{\xi_1^p \xi_2^q \xi_3^r \mid (pqr) \in \mathscr{I}\}$$

where $\mathscr{I}$ represents an indexing set, defined for each element as

$$\begin{aligned}
\mathscr{I}^{\mathrm{quad}} &= \{(pqr) \mid 0 \le p \le P,\ 0 \le q \le Q,\ r = 0\} \\
\mathscr{I}^{\mathrm{tri}} &= \{(pqr) \mid 0 \le p \le P,\ 0 \le p+q \le Q,\ r = 0,\ P \le Q\} \\
\mathscr{I}^{\mathrm{hex}} &= \{(pqr) \mid 0 \le p \le P,\ 0 \le q \le Q,\ 0 \le r \le R\} \\
\mathscr{I}^{\mathrm{pri}} &= \{(pqr) \mid 0 \le p \le P,\ 0 \le q \le Q,\ 0 \le p+r \le P,\ P \le R\} \\
\mathscr{I}^{\mathrm{tet}} &= \{(pqr) \mid 0 \le p \le P,\ 0 \le p+q \le Q,\ 0 \le p+q+r \le R,\ P \le Q \le R\}
\end{aligned}$$

In order to preserve the positivity of discretised spatial operators, we insist that given the components of $\chi = (\chi_1, \dots, \chi_n)$ the determinant of the Jacobian matrix

$$[J_\chi(\xi)]_{ij} = \frac{\partial \chi_i(\xi)}{\partial \xi_j}, \qquad i, j = 1, \dots, n$$

is positive for all $\xi \in \Omega_{\mathrm{st}}$, so that χ preserves orientation and is invertible. Furthermore we consider an isoparametric representation of χ in terms of a set of shape functions ϕ_{pqr}, so that

$$\chi_i(\xi) = \sum_{(pqr)\in\mathscr{I}} (\hat{\chi}_i)_{pqr} \phi_{pqr}(\xi)$$

In the numerical demonstrations below we consider an expansion in terms of a tensor product of modified hierarchical modal functions which permits a boundary-interior decomposition [2]. We note however that in this setting the choice of shape function is relatively unimportant, so long as they span the polynomial space of the element. However, as we will demonstrate later, this choice of basis is useful for certain types of elemental subdivisions as it permits fewer restrictions on the resulting subelement polynomial spaces.

3.1 Subdivision into the same element type

In this section we demonstrate how the isoparametric mapping χ, which we assume has positive Jacobian for all $\xi \in \Omega_{\mathrm{st}}$, can be used to subdivide an element into smaller elements of the same type. The goal of the subdivision process is to obtain a mapping $\zeta : \Omega_{\mathrm{st}} \to \widetilde{\Omega}$ where $\widetilde{\Omega} \subset \Omega$ and $\det J_\zeta(\xi) > 0$ for all $\xi \in \Omega_{\mathrm{st}}$.

In the isoparametric approach we adopt here, instead of attempting to determine the exact subdomain $\widetilde{\Omega}$ of the physical element Ω, we select a subdomain of the standard region, $\widetilde{\Omega_{\mathrm{st}}}$, and construct an invertible mapping $f : \Omega_{\mathrm{st}} \to \widetilde{\Omega_{\mathrm{st}}}$ with $\det J_f(\xi) > 0$. Initially, we also assume that the polynomial expansion in each direction is equal so that $P = Q = R$. Setting ζ as the composition $\chi \circ f$ we then obtain a subelement $\widetilde{\Omega} = \zeta(\Omega_{\mathrm{st}})$.

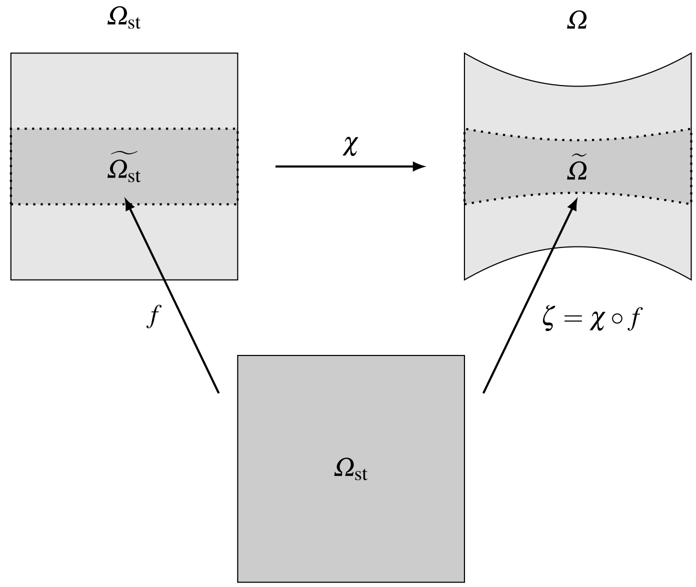

Fig. 2 Construction of the mapping ζ for the subdivision of a quadrilateral element.

The justification for the validity of ζ, and moreover the resulting element $\widetilde{\Omega}$ under the restriction of equal polynomial order is as follows. Firstly, it is clear that the determinant of the Jacobian of ζ is positive for any $\xi \in \Omega_{\mathrm{st}}$, since through an application of the chain rule we have that

$$\det J_{\zeta}(\xi) = \det J_{\chi}(f(\xi)) \det J_f(\xi) > 0. \tag{1}$$

Let us assume that each component of χ lies in the polynomial space $\mathscr{P}(\Omega_{\mathrm{st}})$. In order for ζ to retain the isoparametric representation of the subelements, we note in turn that each of its components must be defined in a polynomial space $\mathscr{P}'(\Omega_{\mathrm{st}})$ where in the most general case, $\mathscr{P}(\Omega_{\mathrm{st}}) \subset \mathscr{P}'(\Omega_{\mathrm{st}})$. A consequence of subdivision therefore is that the subdivided elements may have a higher polynomial order than the parent element depending on the choice of f.

Figure 2 shows a simple application of this subdivision strategy for a quadrilateral element. Here we choose for example an affine mapping $f(\xi_1,\xi_2) = (\xi_1, c\xi_2)$ for some $c \in (0,1)$ so that the standard element is scaled in the ξ_2 direction. Applying the original χ mapping we obtain a new element $\widetilde{\Omega}$ which is appropriately scaled, and naturally introduces curvature into the resulting subelement. In this case, any polynomial term $\xi_1^p \xi_2^q$ is mapped under f to the term $c^q \xi_1^p \xi_2^q$ which clearly lies in $\mathscr{P}(\Omega_{\mathrm{st}}^{\mathrm{quad}})$, and indeed it is clear that by equation 1 that $\det_{\zeta}(\xi)$ is simply a scalar multiple of $\det_{\chi}(\xi)$. We may therefore choose $\mathscr{P}'(\Omega_{\mathrm{st}}) = \mathscr{P}(\Omega_{\mathrm{st}})$ and the order of the subelements may be the same as the parent element.

Since the restriction of equal polynomial order is somewhat restrictive, we now consider the case where the polynomial order in each direction is not equal. Whilst a similar argument to the previous explanation can be used in this case, more care must be taken either in the choice of the mapping f or in the order of the resulting subelements to ensure that the polynomial space is correctly spanned. For example, consider a quadrilateral element with expansion orders $P = 2$ and $Q = 1$ which has corresponding polynomial space $\mathscr{P}$, and suppose we choose to produce a trivially subdivided element by applying the transformation $f(\xi_1, \xi_2) = (-\xi_2, \xi_1)$. This map has positive Jacobian determinant and indeed is affine, as in the previous example. However, since ξ_1 and ξ_2 are permuted in the composition with f, the expansion has polynomial terms which lie outside of $\mathscr{P}$ leading to unpredictable element generation.

There are two solutions in this case. Firstly we may choose to obey the general condition $\mathscr{P}(\Omega_{\mathrm{st}}) \subset \mathscr{P}'(\Omega_{\mathrm{st}})$, and enrich the polynomial order of the subelement so that $P = Q = 2$. Alternatively however, we may permute the polynomial orders of the resulting subelements, so that $P = 1$ and $Q = 2$, to form a space $\mathscr{Q}$. We see in this instance that the resulting subelement is still valid as all of the terms of the original χ expansion are represented in ζ, but the previous condition is not held since $\mathscr{P} \not\subset \mathscr{Q}$. We therefore note that $\mathscr{P}(\Omega_{\mathrm{st}}) \subset \mathscr{P}'(\Omega_{\mathrm{st}})$ represents a sufficient, but not necessary condition on the validity of subelements in this case.

A similar warning also applies to the other element types, and in particular triangles, prisms and tetrahedra since additional conditions are placed on the summation of mode indices which must be observed. In the next section, we discuss a similar enrichment strategy to permit the subdivision of elements into different element types.

3.2 Subdivision into different element types

Another possible strategy one may adopt when subdividing elements is to consider their division into elements of a different type; for instance, we may subdivide a quadrilateral into triangles in two dimensions, or alternatively hexahedra into prisms or prisms into tetrahedra in three dimensions. Such techniques are well understood for linear finite elements [1] but for curvilinear elements self-intersection may occur. In this section we demonstrate how the technique introduced in the previous section can be adapted to introduce curvature into the subelements in such a way as to prevent them becoming invalid.

We must adapt the previous argument above since now $f : \Omega'_{\mathrm{st}} \to \widetilde{\Omega}$ where $\Omega'_{\mathrm{st}} \subsetneq \Omega_{\mathrm{st}}$ and so the polynomial spaces which span these standard elements obey the relation $\mathscr{P}(\Omega'_{\mathrm{st}}) \subsetneq \mathscr{P}(\Omega_{\mathrm{st}})$. In the same way that the technique needs an enrichment of the polynomial space if direction-dependent polynomial orders are used, if we naively apply the method then the polynomial expansion χ can contain terms which are not contained inside $\mathscr{P}(\Omega'_{\mathrm{st}})$, and so the resulting mapping ζ may not produce valid elements.

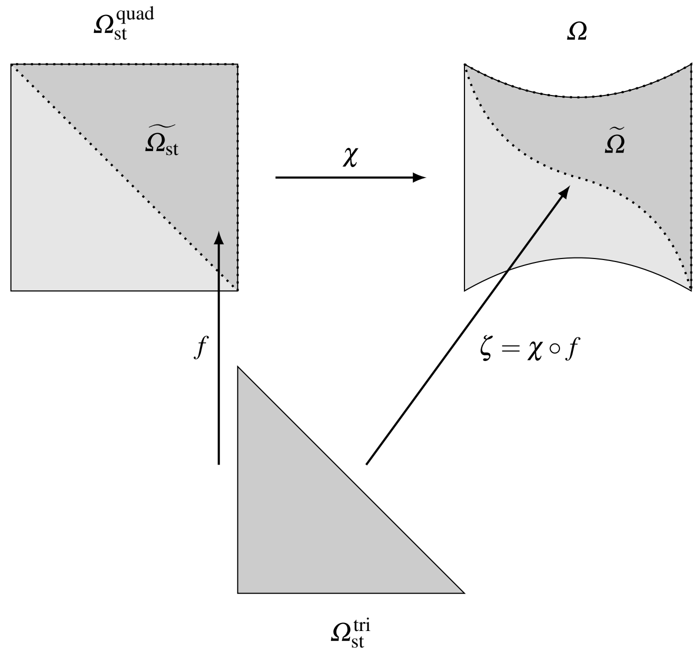

Fig. 3 Construction of the map ζ in the case of a quadrilateral being split into two triangles.

To demonstrate this point, we first examine the problem of figure 3, which depicts an example where a quadrilateral is split along a diagonal edge in order to obtain two triangles. We may again utilise an affine mapping $f(\xi) = -\xi$ in order to map $\Omega_{\mathrm{st}}^{\mathrm{tri}}$ onto a subdomain $\widetilde{\Omega_{\mathrm{st}}}$ of $\Omega_{\mathrm{st}}^{\mathrm{quad}}$. From our previous argument we see that each component of $\zeta = \chi \circ f$ has degree $2P$ in general if the original quadrilateral is of order P.

Since $\zeta \in [\mathscr{P}(\Omega_{\mathrm{st}}^{\mathrm{tri}})]^2$ we must select a sufficiently large polynomial order for the triangular space so that all terms of the expansion are represented in the resulting expansion. To guarantee this for a general quadrilateral-to-triangle split, given a quadrilateral of order P we must generate triangles of order $2P$. Then the space $\mathscr{P}(\Omega_{\mathrm{st}}^{\mathrm{quad}}) \subset \mathscr{P}(\Omega_{\mathrm{st}}^{\mathrm{tri}})$ and thus ζ captures all curvature of the original mapping. For a visual illustration of this, we may represent the polynomial spaces of the triangular and quadrilateral elements in the form of a Pascal's triangle as shown in figure 4.

Figure 5 illustrates the problem of using triangular elements which are not sufficiently enriched. On the left, a second-order ($\mathbb{P}^2$) quadrilateral is split into two second-order triangles. Splitting the quadrilateral into two $\mathbb{P}^2$ triangles leads to the generation of degenerate elements. In this case, the symmetry of the deformed element coupled with the quadratic order of the triangles means that the diagonal edge which bisects the quadrilateral is forced to remain straight and thus causes a self-intersection. We note that in this example, the interior quadrilateral mode $\xi_1^2\xi_2^2$ is not

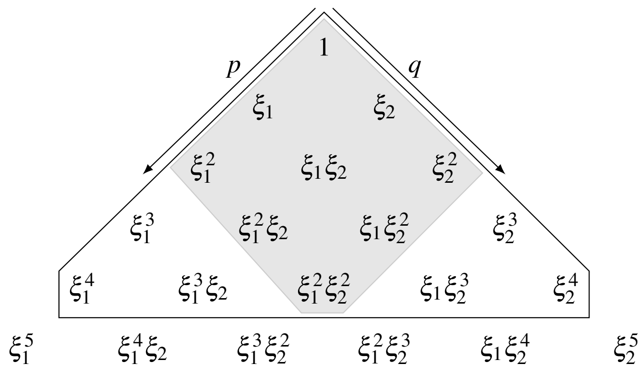

Fig. 4 Pascal's triangle representing the polynomial spaces of $\mathbb{P}^2$ quadrilateral (shaded grey) and $\mathbb{P}^4$ triangular (black outline) elements. The triangle shows that in order to split a general $\mathbb{P}^2$ quadrilateral we require $\mathbb{P}^4$ triangles so that all terms can be represented in the resulting mapping.

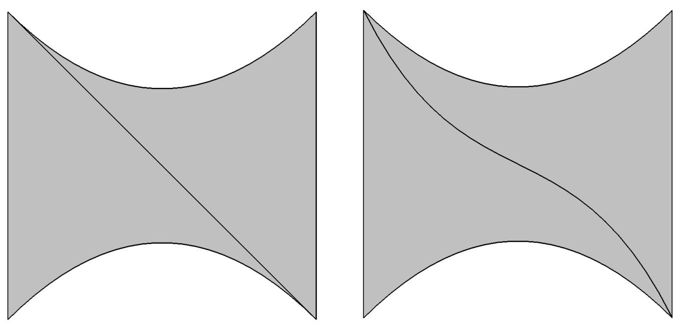

Fig. 5 Qualitative example of the necessary condition for subdivision. A $\mathbb{P}^2$ quadrilateral is split into $\mathbb{P}^2$ (left) and $\mathbb{P}^3$ (right) triangles. Since a $\mathbb{P}^2$ triangular expansion does not capture some of the terms of the original mapping, an additional order is required to produce valid elements.

energised since curvature is only introduced in one coordinate direction. We additionally note that this can be very intuitively achieved by the choice of a boundary-interior hierarchical expansion in which edge and vertex degrees of freedom are decoupled from the interior. Other basis types, such as a nodal Lagrange scheme, will not in general have this property, although the use of the classical Gordon-Hall blending does have this property.

Consulting the Pascal triangle of polynomial spaces we therefore see that only a $\mathbb{P}^3$ expansion is required for the triangular elements. Using this insight, from a qualitative perspective we can predict how the diagonal edge will be deformed under this mapping, since if we consider a parametrisation $r(t) = (t, -t)$ for $t \in [-1, 1]$ then the composition $\chi(r(t))$ should be a cubic polynomial in t. The resulting subdivision, shown on the right-hand side of figure 5, confirms this observation and consequently we obtain two valid triangular elements.

We note that the same logic can be used in the splitting of prismatic and hexahedra elements into tetrahedra. In general an order P prismatic or hexahedral element also requires enrichment so that the resulting tetrahedra have order $2P$ and $3P$ tetrahedra. However by applying the logic above, if curvature is introduced only into the triangular faces of the prisms, then it is only necessary to produce order $P+1$ tetrahedra. Since visualisation of the Pascal's triangle structure is more difficult in three dimensions, this can alternatively be seen from a brief analysis of the prismatic and tetrahedral spaces. If a linear expansion is used in the homogeneous direction of the prismatic element (i.e. $Q=1$) and $P=R$ then the resulting polynomial space is

$$\mathscr{P}^{\text{pri}}(\Omega_{\text{st}}) = \{\xi_1^p \xi_2^q \xi_3^r \mid 0 \le p+r \le P,\ q=0,1\}.$$

A tetrahedron with equal polynomial order P in each direction has the restriction on a triple (pqr) that $0 \le p+q+r \le P$. If $q=1$ then we obtain the restriction $0 \le p+r \le P-1$, and so the tetrahedral space at order P does not contain the prismatic space, leading to possible invalid elements. In order to guarantee validity of elements we therefore require tetrahedra of order $P+1$.

In the following section, we give a demonstration of this prism-to-tetrahedron splitting and also highlight the application of the refinement method in boundary-layer problems.

4 Applications

This section demonstrates the usefulness of the subdivision method by showing how it can be used to generate three-dimensional meshes for challenging applications. Firstly we consider the subdivision of a coarse prismatic boundary-layer mesh into a series of progressively thinner elements as the distance to the wall decreases. We then show how the prismatic elements can be subdivided to obtain a boundary-layer mesh comprising only tetrahedra for use by solvers supporting only simplex elements.

4.1 Boundary layer mesh generation

Figure 6 shows how the subdivision technique can be used to generate a boundary layer mesh for an intercostal pair of a rabbit aorta. In these simulations one wishes to simulate the flow of blood through the aortic arch. From this, one may simulate an advection-diffusion equation in order to measure the concentration of particles which are transported by the flow of blood. Whilst in this case, the Reynolds number of the flow is not particularly large, the diffusion coefficient is inversely proportional to the Peclet number, which in turn is inversely proportional to the size of particle being advected. At high Peclet numbers, in a similar fashion to if the Reynolds

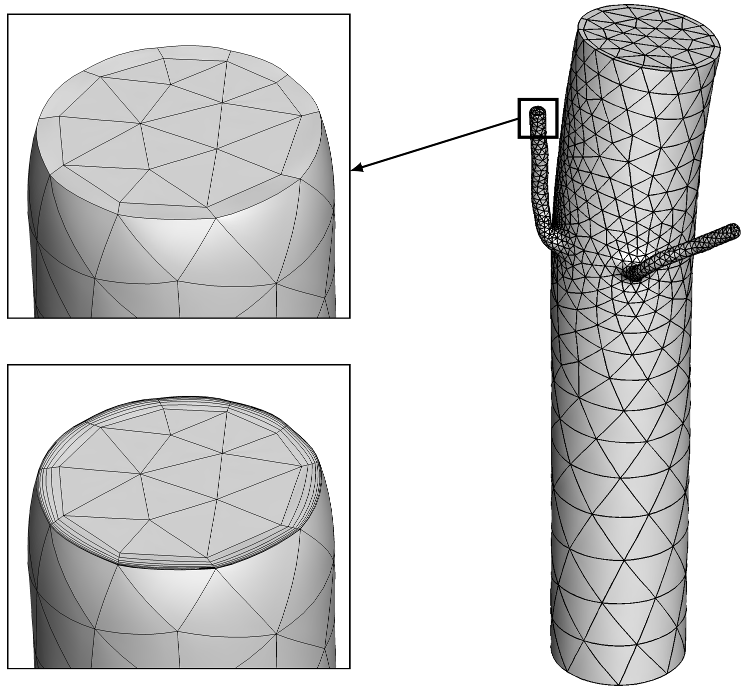

Fig. 6 Boundary layer refinement for an intercostal pair of a rabbit aorta (pictured right). The upper left image shows the high curvature of one smaller vessel, and below the resulting boundary layer mesh is visualised.

number were high, one must use a thin boundary layer in order to resolve the steep gradient of the scalar variable representing concentration.

In order to generate a sequence of N subelements which gradually become more slender towards the surface of the domain, we define a spacing distribution Δ_k for $1 \leq k \leq N$ which defines the height of each prismatic subelement inside $\Omega_{\mathrm{st}}^{\mathrm{prism}}$. In this case, we choose Δ_k to be a geometric progression so that

$$\Delta_k = ar^k, \quad a = \frac{2(1-r)}{1-r^{N+1}}$$

where r denotes a ratio dictating the relative height of each element. Under the framework of section 3 then, we define a straightforward affine scaling function similar to that used in figure 2 which obeys the necessary conditions in order to generate valid subelements. We additionally note that as long as the same spacing distribution is used for all prismatic elements, the resulting mesh is conformal. One of the major advantages of this method for the generation of boundary layer meshes

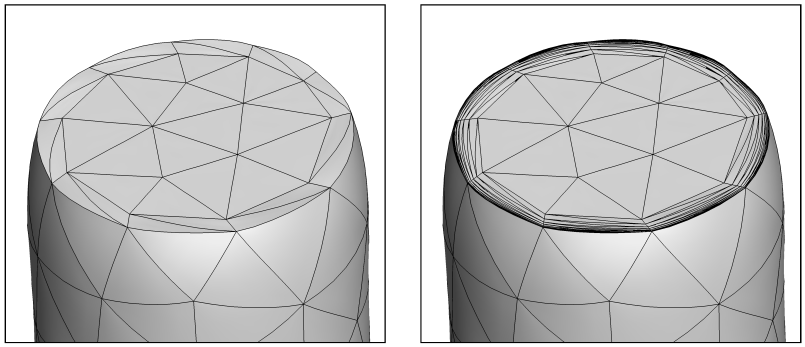

Fig. 7 The result of a tetrahedralisation of the intercostal pair mesh of figure 6. The left-hand figure shows a prism to tetrahedron split of the original mesh; on the right we apply the splitting after the boundary layer refinement has been performed.

is that the resulting subelements are guaranteed to be valid, as shown in section 3, and thus we are able to produce boundary layers of arbitrary thickness.

4.2 Generating meshes of simplex elements

Certain solvers only have support for meshes which are composed only of simplex elements. For problems where boundary layers are required, this poses an additional problem for mesh generation software. In figure 7, we show how the same method can be used to split the prismatic elements of figure 6 into three tetrahedra. Firstly we note that in order for the resulting mesh to be conformal, we must employ a strategy so that the quadrilateral faces which connect prismatic elements are split in a consistent fashion, such as the one outlined by Dompierre et al. [1].

Once this strategy is applied, we may utilise the subdivision strategy to split the standard prismatic element into three tetrahedra by using an affine transformation similar to that used in figure 3. We note that in the specific case of figure 7, since the curvature of the original prisms is only imposed on the triangular surface, we may obtain valid tetrahedra by enriching the polynomial space by one order.

An important point to note is that whilst the validity of the resulting tetrahedra is guaranteed through our previous arguments, this method may lead to the production of tetrahedra which have suboptimal quality in terms of interior angles, depending on the curvature of the original prismatic elements. However, when tetrahedral boundary layers are required this is often unavoidable since the elements are required to possess a large stretching ratio. In the very worst cases, the use of these meshes as a starting point for a mesh deformation procedure may lead to better quality elements. We suggest that the validity of the meshes produced here may lead to improved convergence speeds in such methods.

5 Conclusions

In this paper we have derived the mathematical conditions that are necessary for the subdivision of high-order isoparametric elements, and show how this technique can be applied to tackle challenges in high-order mesh generation. We posit that the simplicity of the method outlined here will prove to be a valuable tool in improving both the efficiency and robustness of curvilinear mesh generation software, and particularly for the generation of meshes for high Reynolds number computational fluid dynamics problems or high Peclet number advection-diffusion problems.

References

1. Dompierre, J., Labbé, P., Vallet, M.G., Camarero, R.: How to subdivide pyramids, prisms and hexahedra into tetrahedra. In: 8th International Meshing Roundtable. Lake Tahoe, California (1999)
2. Karniadakis, G., Sherwin, S.: Spectral/*hp* Element Methods for Computational Fluid Dynamics. Oxford University Press (2005). Second edition
3. Moxey, D., Hazan, M., Sherwin, S.J., Peiró, J.: An isoparametric approach to high-order curvilinear boundary-layer meshing (2014). Under review in Comp. Meth. App. Mech. Eng.
4. Persson, P.O., Peraire, J.: Curved mesh generation and mesh refinement using Lagrangian solid mechanics. In: 47th AIAA Aerospace Sciences Meeting and Exhibit. Orlando (FL), USA (2009). AIAA paper 2009–949
5. Sherwin, S., Peiró, J.: Mesh generation in curvilinear domains using high-order elements. International Journal for Numerical Methods in Engineering **53**(1), 207–223 (2002)
6. Toulorge, T., Geuzaine, C., Remacle, J.F., Lambrechts, J.: Robust untangling of curvilinear meshes. Journal of Computational Physics **254**, 8–26 (2013)
7. Xie, Z., Sevilla, R., Hassan, O., Morgan, K.: The generation of arbitrary order curved meshes for 3D finite element analysis. Computational Mechanics **51**(3), 361–374 (2013)

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